WATER ENTRY AND RELATED PROBLEMS

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ABSTRACT

This thesis is concerned with unsteady problems in fluid dynamics which involve a free surface. The fluid is assumed to be inviscid and incompressible and in all cases a similarity law is applicable, which reduces by one the number of independent variables in the problem. The first chapter is a general introduction. In the second chapter the relevant equations of motion are derived and the existing literature is surveyed. In particular two papers published in 1969, one by Z.N. Dobrovol'skaya and the other by A.G. Mackie are described in some detail as the results are used in subsequent chapters. Chapter 3 is concerned with the application of Lagrangian variables to the water entry and other problems. The equations of motion are derived from this point of view and similarity variables are introduced to reduce the number of independent variables from three to two. Boundary conditions for the wedge water entry problem are set up and the equations of motion are reduced to a quasi-linear first order system. The question of the contact angle between the free surface and the wedge is discussed. It is shown how the equations of motion may be reduced to a single third order equation for an auxiliary function and how this function is related to various geometrical properties of the subsequent motion. The fourth chapter is concerned with a local analysis near the contact point of the free surface and the wedge, and also with the behaviour near the tip of the jet in/

in the aperture problem. This latter problem is an axisymmetric problem in which a jet grows according to a similarity law of the same type as occurs in the water entry problem, and a linearised solution is also given for the aperture problem. Chapter 5 considers a linearised theory for the wedge water entry problem and Chapter 6 is entirely concerned with various aspects of the cone water entry problem. There are three appendices which give the mathematical details of the flow near the tip of the wedge, the Lagrangian representation of the flow due to a dipole, and a second term in the expansion for the flow near the contact point of the free surface and the wedge.

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CONTENTS

CHAPTER 1.

Over recent years considerable interest has been shown in the problem of water entry of a wedge and other related problems. These are all unsteady problems involving a free surface and generally only the case of an inviscid, incompressible fluid has been considered. The unifying feature of the problems which will be discussed is that a similarity transformation is available which reduces by one the number of independent variables. Take, for example, the case of an infinite wedge which plunges at constant speed into a half-space of liquid initially at rest. Then, provided the effects of gravity and surface tension can be ignored, there is no characteristic length or time scale in the problem and the independent variables X, Y and t only occur in the combinations X/t and Y/t so that the number of independent variables has been reduced from three to two. It is to be expected that for an essentially incompressible fluid the effects of gravity and viscosity will be negligible over some period of time, and that surface tension effects may also be neglected.

The main difficulties which arise in these problems are connected with the free surface. If the motion starts from rest it is irrotational and so a velocity potential exists and satisfies Laplace's equation. Although the governing equation is linear and the number of independent variables may be reduced by use of a similarity law, the problems are still far from simple. Instead of a standard boundary value problem for Laplace's equation, one part/

part of the domain, the free surface, is an unknown surface on which two boundary conditions, the dynamic condition of constant pressure and the kinematic condition that particles initially on the boundary stay there, have to be imposed. Rather than having one boundary condition on a known surface, we have two boundary conditions on an unknown surface and it is to be expected that the form of this surface will be determined as part of the solution to the problem. Of course in steady two-dimensional flow the standard method of dealing with an unknown free surface is to regard the velocity potential and stream functions as independent variables and the space variables X and Y as dependent variables so that the free surface simply becomes a straight line in the plane of the new independent variables. Unfortunately this procedure is not possible for unsteady problems, even when limited to the present quasi-steady type.

The first important contribution made to the water-entry problem was by H. Wagner (1932). He proved the significant result that for a wedge entering a half-space of water at constant speed and under a similarity hypothesis, the distance, measured along the free surface, between any two particles which remain on the free surface throughout the motion is a constant in time. An approximate solution for the cone water-entry problem was given by M. Shiffman and D.C. Spencer (1951). Later work was done on the wedge problem by Z.N. Dobrovol'skaya (1964 and 1969), P.R. Garabedian (1953 and 1965) and A.G. Mackie (1962 and 1969). In the next chapter this work is discussed in more detail/

detail and in particular the results of the two 1969 papers are described at some length, since they are relevant to what follows in subsequent chapters. One topic which has been treated in considerable detail by these authors is the question of the contact angle which the free surface makes with the wedge face. It is clear from similarity arguments that this angle is constant in time and it has been shown by various methods that it must be restricted to the range $(0, \pi/4)$.

The main feature of the present work is to approach the water entry problem from a different point of view. J.J. Stoker (1957) has tackled the problem of the breaking of a dam using a Lagrangian rather than an Eulerian representation, and has demonstrated the advantages of this method for unsteady problems which involve a free surface. When Lagrangian variables are used, the displacement of a particle is regarded as a function of its initial position and the time. Since the initial position of a particle is known, the domain in which the solution of the equations is sought is fixed and there is no need to consider the unknown curve which constitutes the free surface. In the case of the water entry problem it is possible to incorporate a similarity transformation to reduce the number of independent variables by one. The consequences of using Lagrangian coordinates combined with similarity are investigated in some detail and although no complete analytical solution is possible, many interesting results emerge, particularly with regard to the contact angle which the free surface makes with the wedge.

Later/

Later chapters of this work are concerned with diverse aspects of the wedge water entry problem and similar problems. A local analysis is carried out which shows that the fact that the contact angle is less than $\pi/4$ is essentially a local property. There is a section on the aperture problem which is an axisymmetric problem in which a jet of water grows according to a similarity law of the same type as is present in the water entry problem. A chapter is devoted to a linearised theory for the water entry of a thin wedge. The final chapter is concerned with the water entry problem for a cone. When approached from the point of view of Lagrangian variables, this problem, although more complicated, is very similar to the two-dimensional wedge problem. Most of the previous work on the water entry problem, which has utilised an Eulerian formulation, has depended largely on the use of complex variable methods and so the results cannot be applied to the three-dimensional problem. It is an additional advantage of employing Lagrangian variables that their use is not restricted in this way.

CHAPTER 2.

(2.1) Equations of motion for unsteady two-dimensional flow with a free surface.

Consider the two-dimensional motion of an inviscid, incompressible fluid. If the motion starts from rest, then the flow is irrotational so that a velocity potential $\Phi(X,Y,t)$ exists, where X and Y are Cartesian coordinates and t is the time. Then Φ satisfies

$$\frac{\partial^2\Phi}{\partial X^2} + \frac{\partial^2\Phi}{\partial Y^2} = 0 \qquad (2.1.1)$$

throughout the fluid.

On any solid boundaries the normal velocity is zero (relative to the boundary) since the fluid is inviscid. If there is a free surface bounding the fluid, given by $Y = F(X,t)$ say, where F is unknown, then on $Y = F(X,t)$ there are two boundary conditions, a kinematic and a dynamic condition. The kinematic condition may be written as

$$\frac{\partial\Phi}{\partial Y} = \frac{\partial F}{\partial t} + \frac{\partial\Phi}{\partial X}\frac{\partial F}{\partial X} \ . \qquad (2.1.2)$$

The dynamic condition expresses the fact that the pressure is known on the free surface, and is

$$\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}(\nabla\Phi)^2 + \frac{\gamma}{\rho R} + gY = 0 \qquad (2.1.3)$$

where γ is the surface tension, R the radius of curvature of the free surface and gravity is the only body force acting on the fluid.

(2.2)/

(2.2) Simplifying role of similarity variables.

In some cases the use of similarity variables may lead to a considerable simplification of the problem. Suppose X,Y,t only occur in the combinations X/t^n, Y/t^n for some positive integer n. This will be the case if there is no characteristic length or time scale in the problem.

Let $X = \alpha_n t^n x$ and let $Y = \alpha_n t^n y$,

where x and y are now dimensionless quantities. For example, if $n = 1$ α_1 has the dimensions of velocity, or if $n = 2$ α_2 has the dimensions of acceleration. Since $\Phi \sim \alpha_n{}^2\, t^{2n-1}$ we may write

$$\Phi(X,Y,t) = \alpha_n{}^2\, t^{2n-1}\, \phi(x,y).$$

Clearly Laplace's equation is unchanged by this transformation and can now be written as

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 .$$

We also have $F(X,t) = \alpha_n\, t^n\, f(x)$ for a dimensionless function f.

Then $$\frac{\partial F}{\partial X} = \frac{df}{dx}$$

$$\frac{\partial^2 F}{\partial X^2} = \frac{1}{\alpha_n\, t^n}\,\frac{d^2 f}{dx^2}$$

$$\frac{\partial F}{\partial t} = n\,\alpha_n\, t^{n-1}(f - xf').$$

Also $$\frac{\partial\Phi}{\partial t} = \alpha_n{}^2\, t^{2n-2}\left\{(2n-1)\phi - n(x\phi_x + y\phi_y)\right\}$$

$$\frac{\partial\Phi}{\partial X} = \alpha_n\, t^{n-1}\,\phi_x$$

$$\frac{\partial\Phi}{\partial Y} = \alpha_n\, t^{n-1}\,\phi_y .$$

For any value of n, (2.1.2) becomes, on $y = f(x)$,

$$\Phi_y - ny = f'(x)\,(\Phi_x - nx) \qquad (2.2.1)$$

The/

The dynamic condition on the free surface, given by (2.1.3) can only be satisfied under special conditions. Suppose $n = 1$, $\alpha_1 = V_o$ where V_o is a typical velocity. It has already been assumed that there is no time-scale in the problem and so the effects of gravity and surface tension must be negligible since the introduction of either would produce an unwanted time-scale. Then (2.1.3) becomes

$$\tfrac{1}{2}(\phi_x{}^2 + \phi_y{}^2) - x\phi_x - y\phi_y + \phi = 0 \quad \text{on} \quad y = f(x) . \qquad (2.2.2)$$

Another possibility is $n = 2$, $\alpha_n = g$. In this case (2.1.3) is

$$\tfrac{1}{2}(\phi_x{}^2 + \phi_y{}^2) - 2(x\phi_x + y\phi_y) + 3\phi + y = 0 . \qquad (2.2.3)$$

Once again the effects of surface tension have to be negligible. Moreover in this problem it is impossible to have a typical velocity since this together with gravity would generate a time-scale. It is possible to accommodate a typical acceleration other than gravity however since their quotient is just a dimensionless parameter which still allows the use of the similarity variables. Although it is possible to introduce into the equations values of n greater than two it is very unlikely that any physical problem could be treated by such a method since the restrictions would be enormous. In the two cases cited above enough simplifying assumptions have had to be made to make the application of this approach very limited.

(2.3) Typical problems.

The water entry problem. An infinite solid wedge plunges with constant speed V_o into a half-space of liquid which is initially at rest at zero pressure. It is assumed that the effects of surface tension and gravity are negligible, at least for some time and then a similarity solution is feasible with $x = X/V_o t$, $y = Y/V_o t$ and $\phi = \Phi/V_o t$. Most of the existing literature is concerned only with the symmetric problem in which the wedge moves vertically downward (see Figure 2.3.1). It is possible also to consider unsymmetric problems in which the wedge enters at an angle inclined to the vertical but these are much more difficult.

The impact problem. This is the converse of the water entry problem. A wedge of liquid moving with constant velocity V_o impinges on a plane wall. Again, one can look for a solution in which $X = V_o t\, x$, $Y = V_o t\, y$, $\Phi = V_o\, t\phi$, under the same assumptions concerning gravity and surface tension (see Figure 2.3.2). It is also possible to consider a generalisation of the first two problems in which a solid and a liquid wedge meet tip to tip with a constant relative velocity V_o (see Figure 2.3.3).

Axisymmetric versions of these three problems may also be considered, with the wedges replaced by cones. The only change in the governing equations is that Laplace's equation now becomes

$$\frac{\partial^2\phi}{\partial x^2} + \frac{1}{x}\frac{\partial\phi}{\partial x} + \frac{\partial^2\phi}{\partial y^2} = 0$$

where $x = 0$ is the axis of symmetry.

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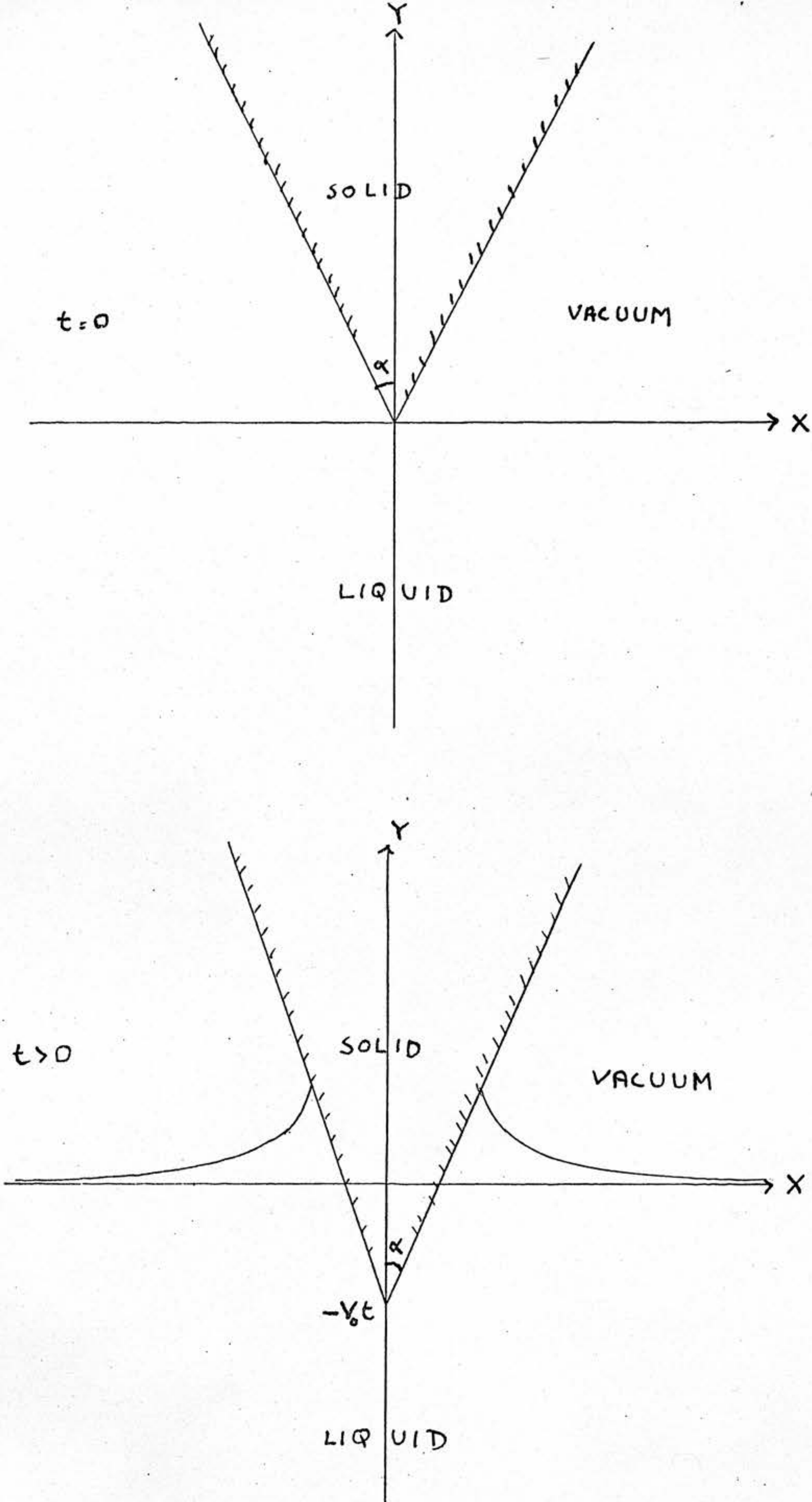


FIG. 2.3.1

Water entry of a wedge

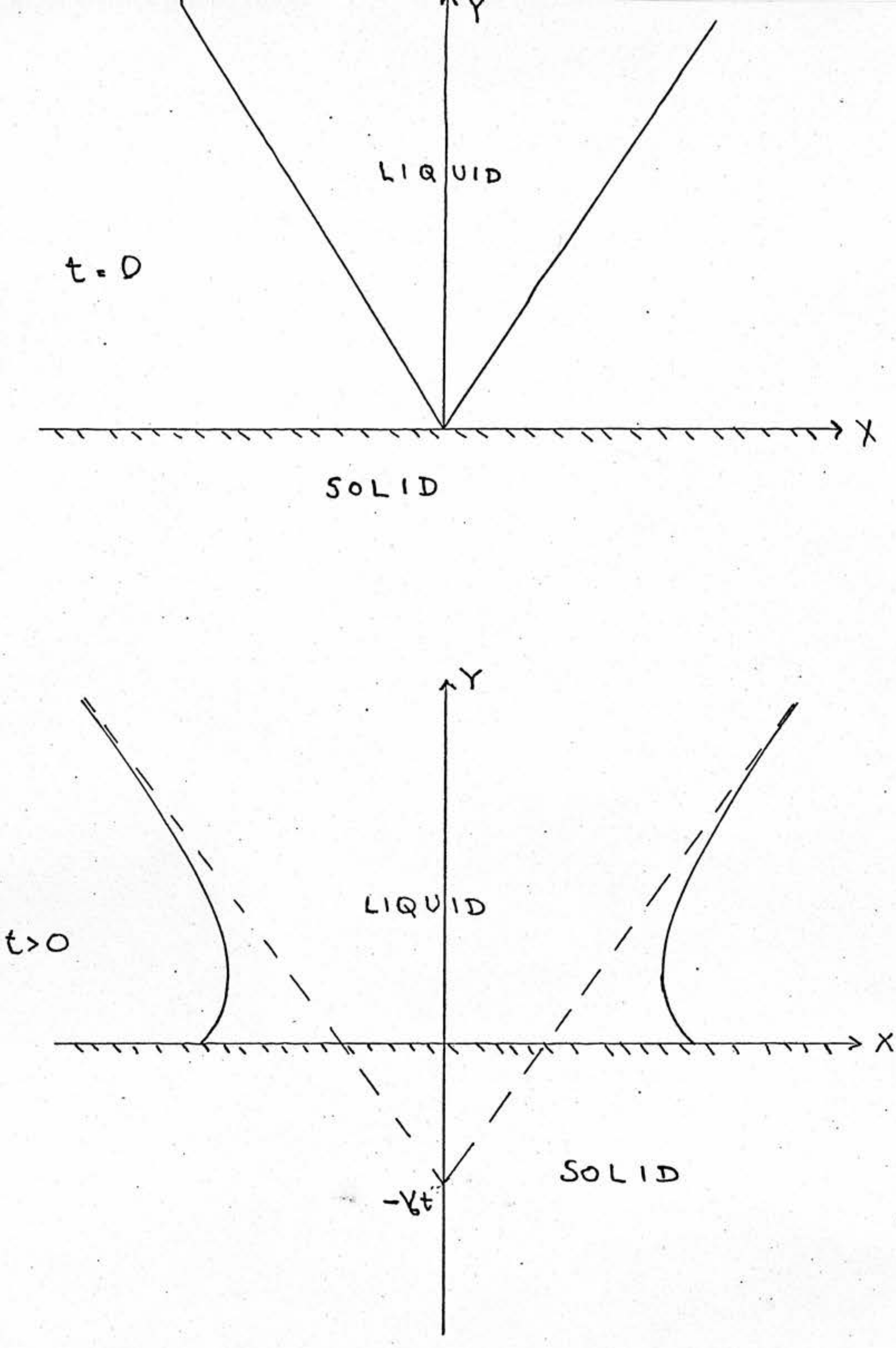


FIG. 2.3.2

The impact problem

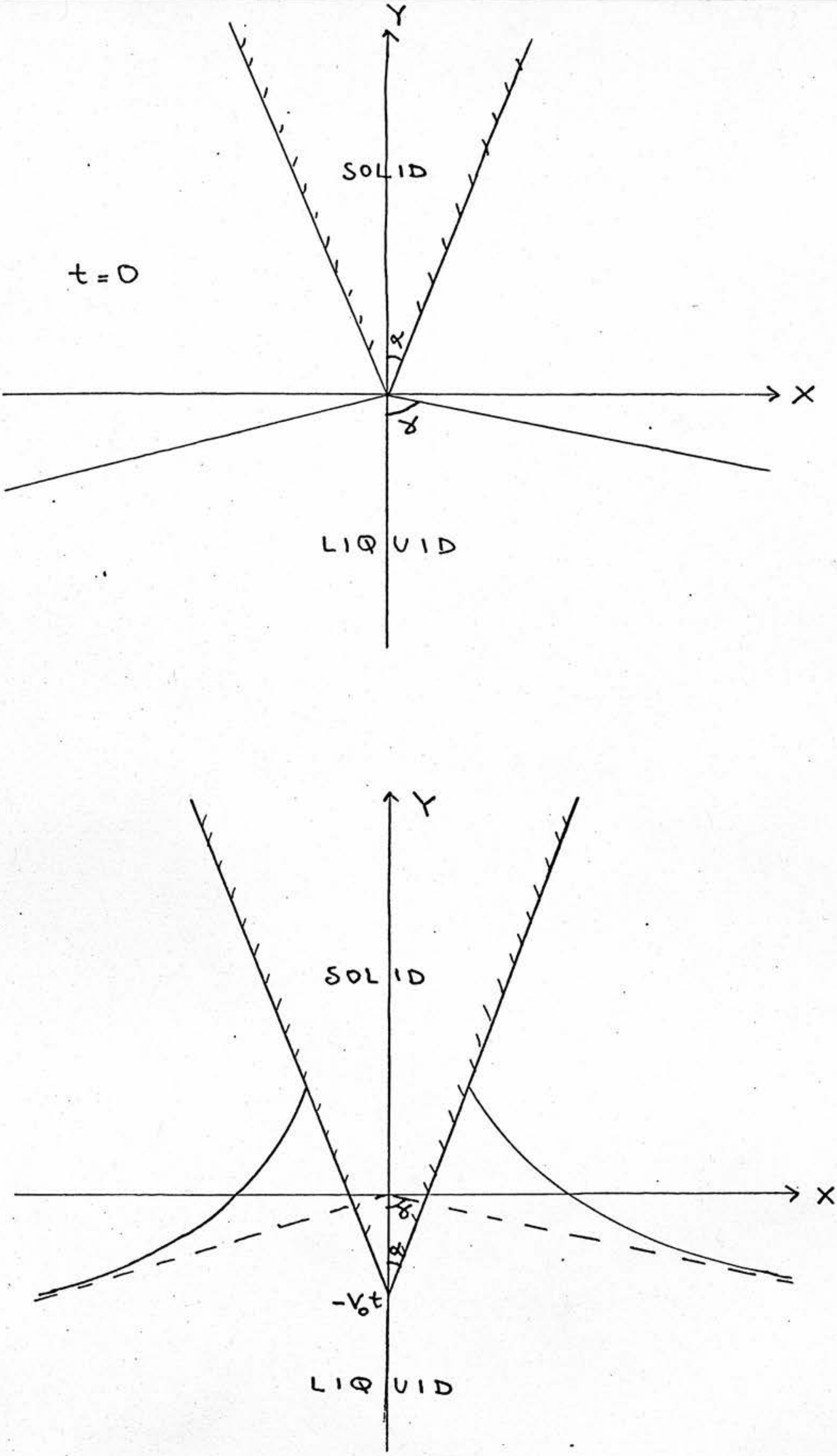


FIG. 2.3.3

Generalised water entry problem

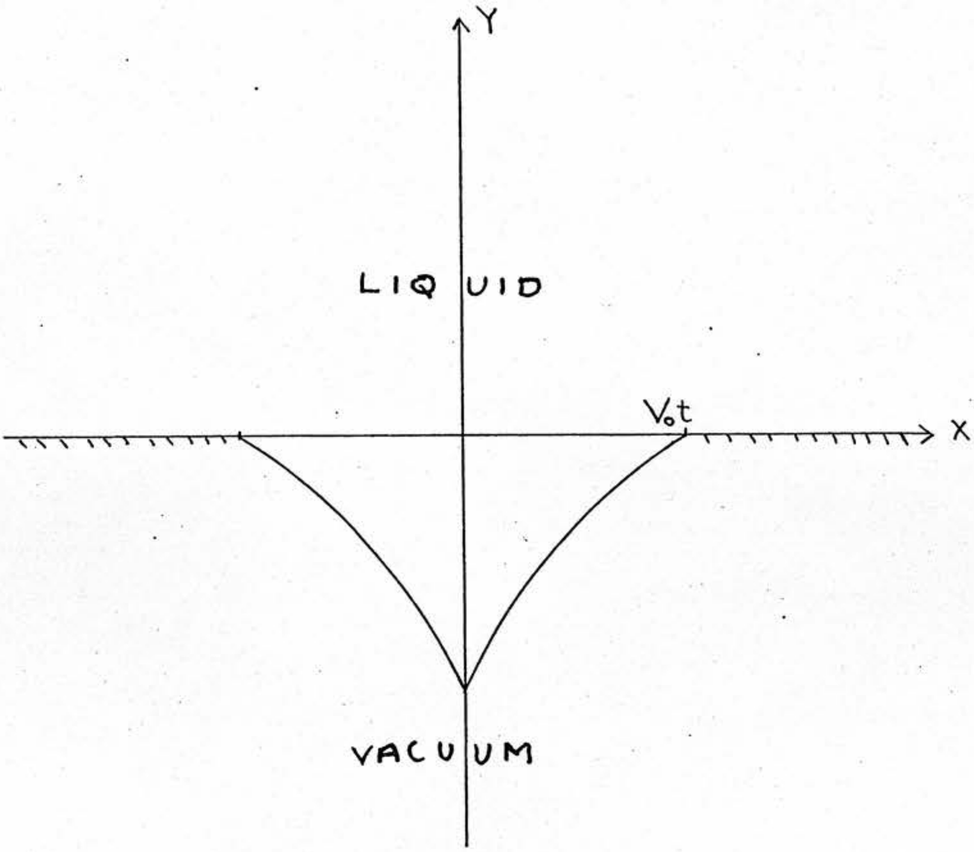


FIG. 2.3.4

The aperture problem

The aperture problem. This is a problem in axisymmetric flow. Liquid is confined at constant pressure in the half-space $Y \geqslant 0$. At time $t = 0$ a circular hole begins to grow so that its radius at time t is $V_o t$ for constant V_o, and an axisymmetric jet of liquid flows out of the hole according to a similarity law of the type already described (see Figure 2.3.4).

(2.4) Survey of existing literature.

There is a considerable body of literature concerned with the water entry problem and a lesser amount with the other problems mentioned in the previous section. As far as the water entry problem is concerned, much interesting work has been done by H. Wagner (1932), P.R. Garabedian (1953 and 1965), Z.N. Dobrovol'skaya (1964 and 1969) and A.G. Mackie (1962 and 1969). The object of the present section is to give a short survey of these works with particular reference to results which will be needed in subsequent chapters.

The notation for this chapter is illustrated in Figure (2.4.1). In general, capital letters will denote the dimensional variables and small letters their dimensionless counterparts.

Wagner (1932) gives an approximate solution to this problem in which he employs the similarity variables. His more important contribution, however, is the introduction of the so-called Wagner function which plays an important part in the subsequent theories.

Let $z = x + iy$. Let $\phi(x,y)$ and $\psi(x,y)$ be the dimensionless velocity potential and stream functions respectively. Let $\zeta(z) = \phi(x,y) + i\,\psi(x,y)$. Wagner's function $h(z)$ is then defined to be

h/

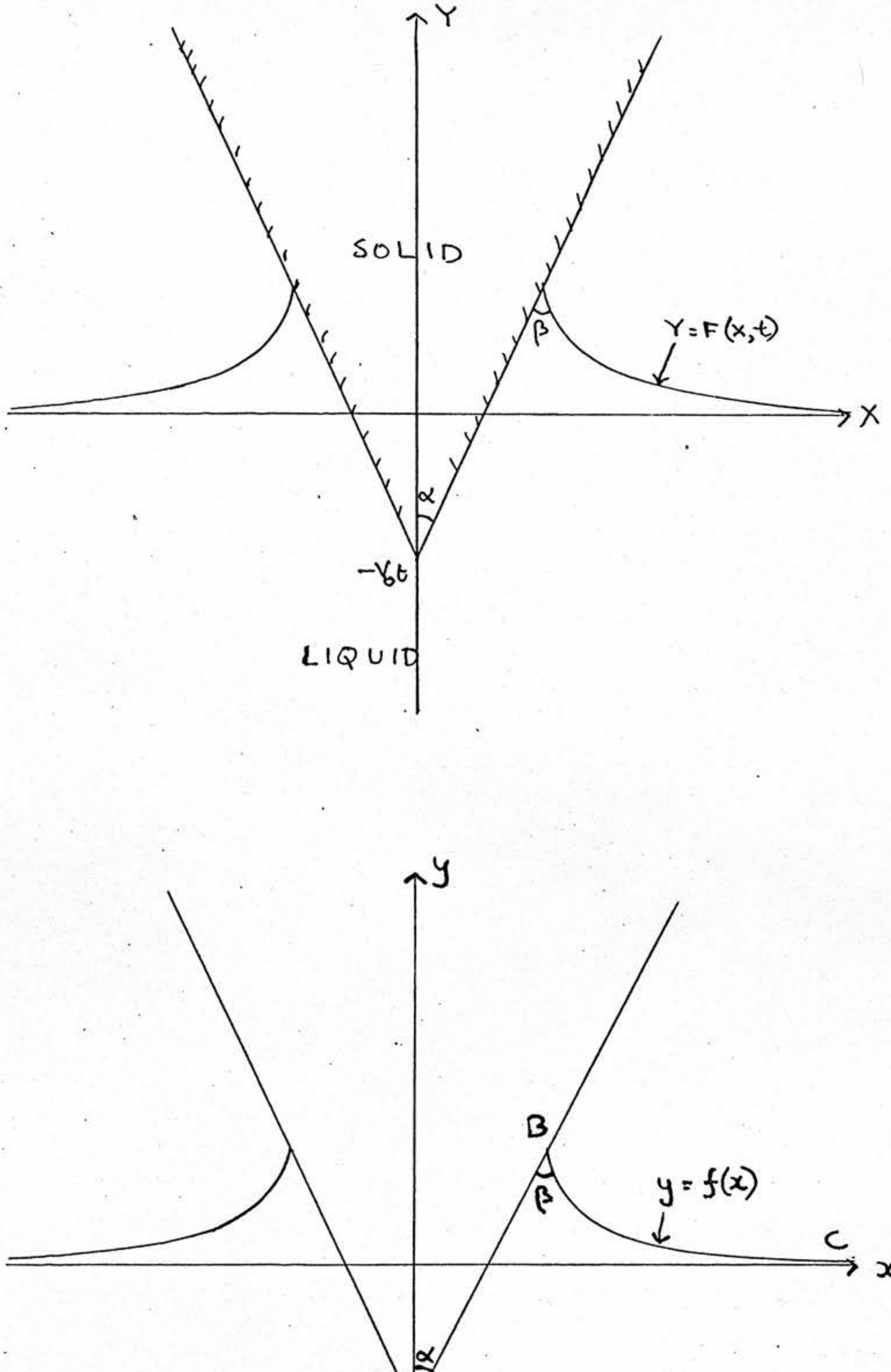


FIG. 2.4.1

Water entry of a wedge

$$h(z) = \int_{\infty}^{z} \left(\frac{d^2\zeta}{dz^2}\right)^{\frac{1}{2}} dz \; . \qquad (2.4.1)$$

The importance of this function is that in the h-plane the boundary of the flow region ABCD becomes a finite triangle. Wagner shows correctly that the boundary in the h-plane consists of straight line segments, but fails to realise that C and D are mapped into the same point and that the resultant figure is bounded. For the water-entry problem the flow region maps into the interior of an isosceles triangle (see Figure 2.4.2). The details of this mapping may be found in Mackie (1969) or Dobrovol'skaya (1969) but it follows fairly simply from consideration of $d^2\zeta/dz^2$ on each portion of the boundary in turn. Wagner also proves the conservation of arc-length property - that the distance between any two particles on the free surface remains invariant throughout the motion, distance being measured along the free surface.

In 1964 Dobrovol'skaya took up this problem and developed it considerably, making extensive use of complex variable theory. In 1969 she carried the method further and gave a complete numerical solution of the water entry problem. All the important results of the 1964 paper are embodied in the later one and so the results which follow are quoted from the 1969 paper.

From section (2.2) we know that the problem for $\phi(x,y)$ is to solve $\nabla^2\phi = 0$ subject to

$$\frac{\partial\phi}{\partial x}\cos\alpha - \frac{\partial\phi}{\partial y}\sin\alpha = \sin\alpha \qquad \text{on AB} \qquad (2.4.2)$$

$$\frac{\partial\phi}{\partial x} = 0 \qquad \text{on AD} \qquad (2.4.3)$$

$$\frac{\partial\phi}{\partial y} - y = f'(x)\left(\frac{\partial\phi}{\partial x} - x\right) \qquad \text{on BC} \qquad (2.4.4)$$

$$\tfrac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \tfrac{1}{2}\left(\frac{\partial\phi}{\partial y}\right)^2 - x\frac{\partial\phi}{\partial x} - y\frac{\partial\phi}{\partial y} + \phi = 0 \quad \text{on BC} \; . \qquad (2.4.5)$$

First/

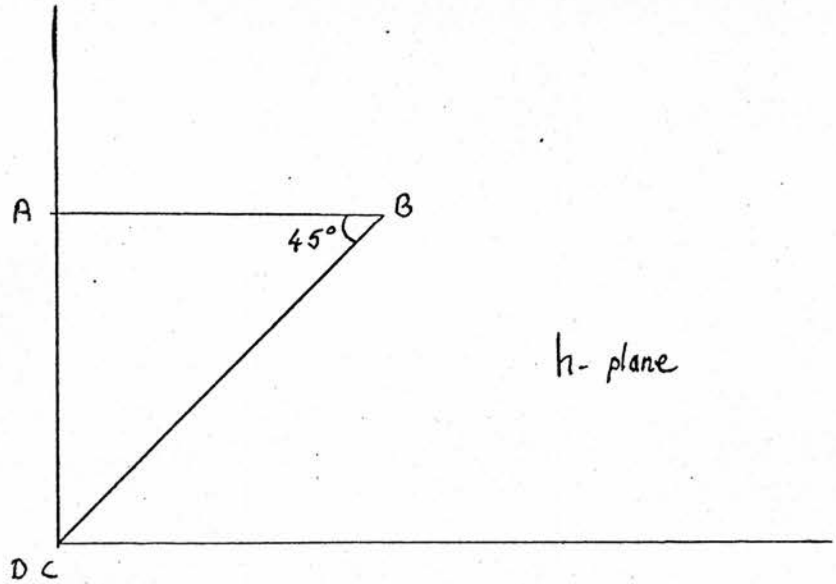


FIG. 2.4.2

The flow region in the plane of Wagner's function

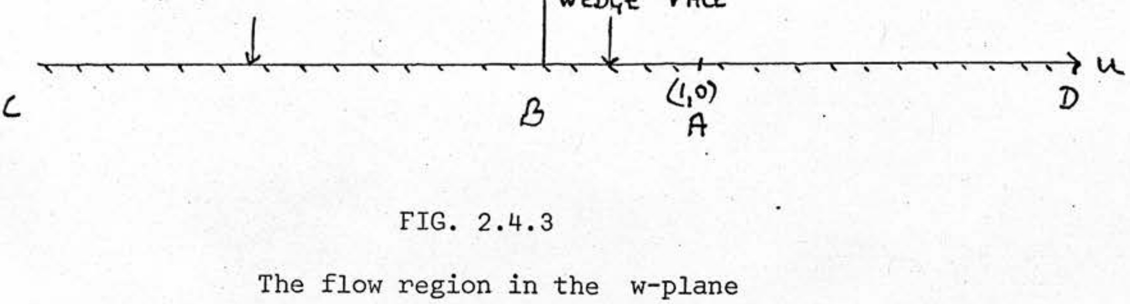


FIG. 2.4.3

The flow region in the w-plane

First of all Dobrovol'skaya reduces the problem to that of finding two functions analytic in the upper half plane. Let $\zeta = \phi + i\psi$ be the complex potential. Let $w(z) = u + iv$ be an analytic function which maps the flow region in the z-plane conformally onto the upper half w-plane so that the point B corresponds to the origin and A to $(1,0)$. (See Figure 2.4.3).
The boundary conditions (2.4.2)-(2.4.5) may be written in terms of $\zeta(w)$ and $z(w)$. Dobrovol'skaya next proceeds to reduce the problem to that of finding a single function $z(w)$ by making use of Wagner's function, The function $h(w)$ which maps the triangle ABCD in the h-plane conformally onto the upper half w-plane is simply a Schwarz-Christoffel transformation. The functions h and ζ can then be eliminated and the problem is to find $z(w)$ subject to three conditions which hold on the three parts of the real w-axis. It so happens that $\arg z'(u)$ is known for $0 \leqslant u < \infty$. The trick is to let $\arg z'(u) = -\pi \{f(u)+1\}$ for $-\infty < u \leqslant 0$. Then $f(u)$ is essentially the angle which the free surface makes with the horizontal. Moreover $z'(w)$ can be written, by means of a Schwarz integral formula for the upper-half plane, in terms of $\arg z'(u)$, and hence of $f(u)$. Substitution of the value of $z'(u)$ obtained in this way into the boundary condition which holds on $-\infty < u \leqslant 0$, and is just the condition that the pressure is zero on the free surface, leads to the following non-linear singular integral equation for $f(u)$.

$$f(u) = -\frac{1}{\pi}\frac{c_o^{\,2}}{c^2}\int_{-\infty}^{u} \frac{u^{-1-\alpha/\pi}(u-1)^{-1+\alpha/\pi}\exp\left[\int_{-\infty}^{0}\frac{f(u_1)du_1}{u_1 - u}\right]}{\int_0^u iu^{-\frac{1}{2}+\alpha/\pi}(u-1)^{-\alpha/\pi}\exp\left[-\int_{-\infty}^{0}\frac{f(u_1)du_1}{u_1 - u}\right]}\,du\,. \tag{2.4.6}$$

The/

The quotient c_o^2/c^2 is essentially a known parameter. Once this equation is solved it is a comparatively straightforward matter to deduce the mapping function $z(w)$, the complex potential $\zeta(w)$, the equation of the free surface, the pressure distribution and any other quantity that may be required.

Dobrovol'skaya is able to show that the integral on the right-hand side of (2.4.6) converges if and only if the contact angle β is such that $\beta < \pi/4$. She then solves equation (2.4.6) by constructing a sequence of successive approximations which are convergent. The numerical calculations are carried out for a variety of wedge angles α ranging from $\alpha = 0{\cdot}018^\circ$ to $\alpha = 60^\circ$. The results thus obtained are interesting but contain no major surprises. The free surface curves are all found to be convex to the fluid and the splash height - the distance of the contact point above the initial level of the free surface - increases as the wedge angle increases. The contact angle β decreases from 18° to 0° as α increases from 0° to 90°. The pressure is positive on the wedge face and for small values of α reaches its maximum at the tip of the wedge whereas for larger values of α the maximum is attained at a point above the initial level of the free surface. As a check on the accuracy of the numerical results, Dobrovol'skaya shows that the condition of conservation of arc-length on the free surface is satisfied to within acceptable bounds. Dobrovol'skaya claims that there would be no great difficulty in increasing the accuracy of these results and extending the range of values of α for which they can be carried out. It should also be possible to apply Dobrovol'skaya's method to related problems, such as the two-dimensional problems mentioned in section (2.3).

Mackie's/

Mackie's first paper (1962) on the water entry problem is concerned with the linearised theory. It is easy to show that, for small α , the linearised water entry problem is

$$\nabla^2\phi = 0 \qquad \text{in} \quad x \geqslant 0, \quad y \leqslant 0.$$

$$\frac{\partial\phi}{\partial x} = \alpha \qquad \text{on} \quad x = 0, \quad -1 < y < 0 \qquad \text{(from (2.4.2))},$$

$$\frac{\partial\phi}{\partial x} = 0 \qquad \text{on} \quad x = 0, \quad y < -1 \qquad \text{(from (2.4.3))},$$

$$\phi = 0 \qquad \text{on} \quad y = 0, \quad x > 0 , \qquad \text{(from (2.4.5))}.$$

The free surface is given by

$$xf'(x) - f(x) = -\frac{\partial\phi}{\partial y}(x,0), \qquad \text{(from (2.4.4))}.$$

The velocity potential $\phi(x,y)$ may be written down immediately in terms of a Green's function and is

$$\phi(x,y) = \frac{\alpha}{2\pi} \int_0^1 \log \frac{x^2 + (y+\theta)^2}{x^2 + (y-\theta)^2}\, d\theta . \tag{2.4.7}$$

The free surface is then found to be

$$y = f(x) = \frac{\alpha}{\pi}\left\{ \log\left(1 + \frac{1}{x^2}\right) + 2x \tan^{-1}\frac{1}{x} - 2 \right\}. \tag{2.4.8}$$

This linearised solution is valid for small α , except in the neighbourhood of the points (0,0) and (0,-1). In fact the values given by the linearised solution are in close agreement with Dobrovol'skaya's computed results for sufficiently small values of the wedge angle α .

Mackie also gives the linearised solution for the water entry of a cone. This may be found using slender body theory. If x and y are now cylindrical polar coordinates with $x = 0$ as the axis of symmetry, then

ϕ /

$$\phi(x,y) = \frac{\alpha^2}{4\pi} \{-\int_{-1}^{0} \frac{2\pi(1+\xi)d\xi}{x^2+(y-\xi)^2} + \int_{0}^{1} \frac{2\pi(1-\xi)d\xi}{x^2+(y-\xi)^2}\} .$$

$$\Rightarrow \phi(x,y) = \frac{\alpha^2}{2} \{\sqrt{x^2+(1+y)^2} - \sqrt{x^2+(1-y)^2} - (1+y)\sinh^{-1}(\frac{1+y}{x})$$

$$+ (1-y)\sinh^{-1} (\frac{1-y}{x}) + 2 \sinh^{-1} (\frac{y}{x}) \} . \qquad (2.4.9)$$

It is easily verified that ϕ is a harmonic function and satisfies the following boundary conditions

$\phi(x,0) = 0$ for $x > 0$,

$\frac{\partial\phi}{\partial x}(0,y) = 0$ for $y < -1$,

$\frac{\partial\phi}{\partial x}(x,y) - \alpha\frac{\partial\phi}{\partial y}(x,y) = \alpha + O(\alpha^3 \log \alpha)$ on $x = \alpha(1+y)$,

$\phi = O(1/r^2)$ as $r \to \infty$ where $r^2 = x^2 + y^2$.

The free surface is given by

$$y = f(x) = \alpha^2 \{-\sinh^{-1}(\frac{1}{x}) + \sqrt{1+x^2} + \frac{1}{2x} - x\} . \qquad (2.4.10)$$

This solution is not valid on any portion of the line segment $x = 0, -1 \leq y \leq 0$, unlike the linearised solution for the wedge problem. However, the behaviour at infinity is still like that due to a dipole at the origin, which in this case is of course a three-dimensional dipole.

Mackie's later paper (1969) is concerned with various aspects of the full non-linear wedge water entry problem. He re-derives the conservation of arc-length property first proved by Wagner (1932) and certain other relations connected with this.

Let $X(s,t)$ and $Y(s,t)$ be the Lagrangian coordinates of a fluid particle which is initially on the undisturbed free surface and at a distance s from the origin. Let $Z = X + iY$. Since Z/s is a dimensionless function of $s/V_o t$,

z/

$$Z = sZ_s + tZ_t \quad . \tag{2.4.11}$$

Differentiate with respect to t to obtain

$$sZ_{st} + tZ_{tt} = 0 \ . \tag{2.4.12}$$

Now the pressure is zero on the free surface and so the pressure gradient is normal to that surface. Hence the acceleration vector Z_{tt} is also normal to the free surface and so

$$\mathrm{Re}(Z_s \, \overline{Z}_{tt}) = 0 \ .$$

In conjunction with (2.4.12) this implies

$$\frac{\partial}{\partial t} \ |Z_s|^2 = 0 \ .$$

Thus the distance between any two particles on the free surface remains constant throughout the motion.

If ζ is the dimensionless velocity potential, then (2.4.11) may be rewritten as

$$t \frac{d}{dZ} (V_o{}^2 t \, \zeta) = \overline{Z} - s\overline{Z}_s \tag{2.4.13}$$

Differentiate with respect to s to get

$$tZ_s{}^2 \, \frac{d^2(V_o{}^2 t\zeta)}{dZ^2} = -sZ_s \, \overline{Z}_{ss}$$

$$= is \ |k| \ \delta \tag{2.4.14}$$

where $\delta = \begin{cases} +1 & \text{if } \frac{d}{ds} \ (\frac{dY}{dX}) > 0 \\ -1 & \text{if } \frac{d}{ds} \ (\frac{dY}{dX}) < 0 \end{cases}$

and k is the curvature and is defined to be positive where d^2Y/dX^2 is positive.

Put $t = 1, V = 1$ without loss of generality. Then (2.4.13) and (2.4.14) may be conveniently written as

$$\frac{d\zeta}{dZ} = \overline{Z} - s\overline{Z}_s \quad \text{and} \quad Z_s{}^2 \, \frac{d^2\zeta}{dZ^2} = i \ s \ |k| \ \delta \quad . \tag{2.4.15}$$

These latter results were first proved by Garabedian in 1953.

Mackie next proceeds to examine the validity of the Wagner mapping function. He shows that, in order that the flow region is/

is mapped onto a finite triangle it is necessary that $\zeta'(z)=O(z^{-2})$ as $z \to \infty$, that the velocity is monotonic on DA and on AB and that the free surface is convex to the fluid. The significance of the first of these conditions is that the algebraic sum of source strengths in the finite part of the plane is zero, and this follows from the fact that the pressure is constant on the free surface. The assumption that the velocity is monotonic on AD and AB seems to be a reasonable one. However, the convexity of the free surface is not a simple matter, although most authors pass it over without comment.

Mackie is able to prove the following theorem.

Theorem: If $p = 0$ on the boundary of a free surface in two-dimensional self-similar flow and if $p > 0$ in the interior of the fluid, then the free boundary is convex to the fluid.

Proof: Consider Bernoulli's equation in the form

$$\frac{2p}{\rho V_o^2} + \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 - 2x\,\frac{\partial\phi}{\partial x} - 2y\,\frac{\partial\phi}{\partial y} + 2\phi = 0 \qquad (2.4.16)$$

where p is the (dimensional) pressure. On the free surface $y = f(x)$ the kinematic condition (2.4.4) holds, and

$$\frac{\partial\phi}{\partial y} - y = f'(x)\left(\frac{\partial\phi}{\partial x} - x\right) \qquad (2.4.17)$$

Let $X = \phi - \frac{1}{2}x^2 - \frac{1}{2}y^2$.

Since $\nabla^2\phi = 0$, $\nabla^2\chi = -2$. $\qquad$ (2.4.18)

From (2.4.16),

$$\frac{2p}{\rho V_o^2} + \left(\frac{\partial\chi}{\partial x}\right)^2 + \left(\frac{\partial\chi}{\partial y}\right)^2 + 2\chi = 0 . \qquad (2.4.19)$$

Also (2.4.17) gives

$$\frac{\partial\chi}{\partial y} = f'(x)\,\frac{\partial\chi}{\partial x} \quad \text{on} \quad y = f(x) . \qquad (2.4.20)$$

Now/

Now regard χ as the velocity potential of a fictitious irrotational steady flow in the x,y variables. This flow will not be incompressible because of (2.4.18). However, since the motion is steady, the acceleration vector is given by $(\underline{Q}.\nabla)\underline{Q}$, where $\underline{Q} = \nabla\chi$ is the velocity, and this is $\nabla(\frac{1}{2}Q^2)$ since the flow is irrotational. Moreover from (2.4.20), the curve $y = f(x)$ is a streamline. The gradient of (2.4.19) gives

$$\frac{\nabla p}{\rho V_o^{\,2}} + \nabla(\tfrac{1}{2}Q^2) + \underline{Q} = 0 .$$

Since $p = 0$ on $y = f(x)$ and by hypothesis $p > 0$ in the interior, ∇p points normally into the fluid. The vector $\underline{Q}$ is along the boundary $y = f(x)$ and hence for equilibrium $\nabla(\frac{1}{2}Q^2)$ must point out of the fluid. The curve $y = f(x)$ is therefore convex to the fluid by the same reasoning as establishes this result in two-dimensional steady flow. It is now necessary to show that p is non-negative in the interior of the fluid. Taking the Laplacian of (2.4.16) we see that $\nabla^2 p \leqslant 0$ and so by the maximum principle p takes its minimum on the boundary of the domain. For the water entry problem it is only necessary to show that p is non-negative on the wedge face. There appears to be no way of establishing this result rigourously, although it is a reasonable assumption, and one which is supported by the linearised theory.

Mackie next considers the question of the contact angle which the free surface makes with the wedge. By using complex variable methods he is able to show that this angle β lies between 0 and $\pi/4$. Moreover, depending on whether the curvature at the contact point is infinite, finite but non-zero, or zero, then β is greater than/

equal to, or less than $\pi/6$. If $\alpha \gtrsim \pi/3$, then $\beta < \pi/6$ and the curvature is zero. These results follow from consideration of the behaviour of the mapping function $w(z)$ near the contact point. A further property, which follows from the convexity of the free surface is that β satisfies $\alpha + \beta \lesssim \pi/2$. This is a significant fact and will be discussed in more detail later.

Garabedian's two papers are concerned with more general aspects of water entry and free surface problems than have been considered up till now. The first paper (1953) is about the water entry of a wedge, but the problem is an unsymmetric one with a pressure difference on either side of the wedge at the moment of impact. The results with the most relevance to the present problem are those which have already been mentioned (2.4.11)-(2.4.15). It is also interesting that the free surface makes an angle with the wedge of $\pi/2$ on one side of the wedge and π on the other. The second paper (1965) is entitled 'Asymptotic description of a free boundary at the point of separation.' Garabedian considers self-similar flow with a free boundary and makes extensive use of complex variable theory. He obtains an integral equation for a mapping function, whose solution would completely determine the water-entry problem, and other allied problems. Lacking an analytic solution, he is able nevertheless to find asymptotic solutions which show that, for the water entry problem, the contact angle β is restricted to $0 < \beta \lesssim \pi/4$. It is interesting that the method also suggests the possibility of $\beta = \pi/2$ or π, but these have to be rejected on account of the convexity of the free surface and the consequent fact that $\alpha + \beta \lesssim \pi/2$.

To/

To complete this survey of literature concerned with the problems mentioned in (2.3) it is necessary to consider briefly one paper concerned with the impact problem and another with the water entry of a cone.

Cumberbatch (1960) has written a paper about the impact of a liquid wedge on a plane solid wall (see Figure 2.4.4). He finds a solution valid at large distances from the wall (region I) and another solution valid near the contact point (region II). He matches these two solutions as smoothly as possible and is able finally to obtain a solution for region III by a relaxation method. Numerical calculations are carried out for angles $\alpha = 45^{\circ}$, $22{\cdot}2^{\circ}$. It is then found that the contact angles are 3° and $4{\cdot}8^{\circ}$ respectively. The pressure on the wall reaches its maximum at $x = 0$, remains approximately constant throughout region III, decreases rapidly between regions III and II to a value close to zero, and finally becomes zero at the contact point. The results can be shown to satisfy a mass-conservation property and the conservation of arc-length property to a reasonable degree of accuracy.

Shiffman and Spencer (1951) obtained a detailed approximate solution for the cone water entry problem on the assumption that the contact of the free surface is tangential. In view of what is known about the wedge a considerable amount of doubt must hang over the validity of their results. This is another question about which more will be said later.

(2.5)/

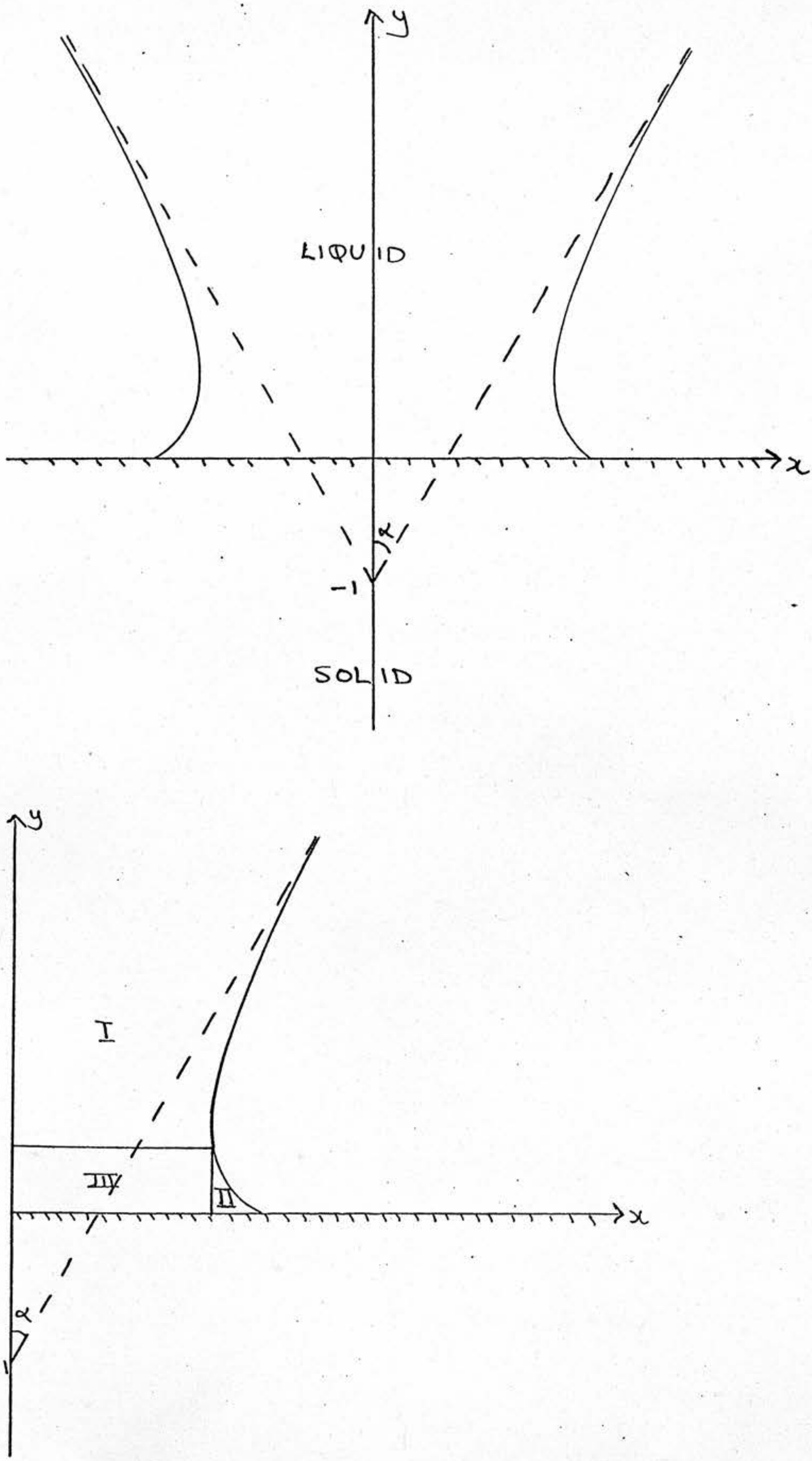


FIG. 2.4.4

Cumberbatch's problem

(2.5) Possibility of a different approach.

All the work mentioned above has been limited in one way or another. The most obvious difficulty connected with the water entry problem is the treatment of non-linear boundary conditions on an unknown free surface. Since the free surface is not a streamline in the similarity variables the usual methods of treating free streamlines in steady flow are not available. The conformal mappings which are introduced to reduce the flow region to a known domain are very effective for the wedge problem but cannot be extended to more practical problems like the cone water entry problem in three dimensions.

One way of reducing the flow region to a known domain is to use a Lagrangian formulation of the problem instead of an Eulerian one. The idea of this method is to find the position of any fluid particle at any time, given its initial position. Since the initial position of each particle is known, the domain in which the equations of motion are to be solved is known and fixed. Of course, as a compensation for this simplification the governing equations turn out to be highly complicated, but there are considerable advantages in this approach, not the least being the possibility of treating three-dimensional problems as well as the two-dimensional ones about which more is already known.

CHAPTER 3.

(3.1) Introduction.

There are two possible methods of formulating problems in fluid mechanics. The more usual method is the Eulerian one, in which quantities such as velocity and pressure are calculated at fixed points in space and time. The Lagrangian approach on the other hand considers the behaviour of individual particles of fluid and in fact calculates the path which a particle follows in space and time. Either of these methods may be preferable in particular problems. The Lagrangian equations of motion become exceedingly difficult in any problems involving viscosity, but the method may be used with advantage in problems of one-dimensional gas dynamics and is particularly useful in problems in classical hydrodynamics involving a free surface, such as the present quasi-steady type of problems.

(3.2) Basic equations.

Consider a two-dimensional problem in inviscid flow. Let the position of any particle of fluid at any time be (X,Y). Suppose that initially $X = A$ and $Y = B$. Then the subsequent position of a particle depends on A,B and the time t so that $X = X(A,B,t)$ and $Y = Y(A,B,t)$. (In fact A and B need not be the initial coordinates of a particle - any two quantities which sensibly identify a particular particle may be used). The acceleration of a particle at time t is thus (X_{tt}, Y_{tt}). By Newton's Second Law of Motion

X/

$$X_{tt} = -\frac{1}{\rho} P_X + F_1 \tag{3.2.1}$$

$$Y_{tt} = -\frac{1}{\rho} P_Y + F_2 \tag{3.2.2}$$

where P is the pressure, ρ the density of the fluid and $\underline{F} = (F_1, F_2)$ the body force acting on the fluid. For simplicity assume that gravity is the only body force acting and write $F_1 = 0$ and $F_2 = -g$. The derivatives of P with respect to X and Y may be eliminated by writing

$P_A = P_X X_A + P_Y Y_A$ and $P_B = P_X X_B + P_Y Y_B$. Multiply (3.2.1) by X_A and (3.2.2) by Y_A and add.

Then $$X_{tt}X_A + Y_{tt}Y_A = -P_A/\rho - g\, Y_A . \tag{3.2.3}$$

Similarly $$X_{tt}X_B + Y_{tt}Y_B = -P_B/\rho - g\, Y_B . \tag{3.2.4}$$

To derive a continuity equation, consider a small element of fluid initially rectangular, with corners (A,B), $(A+\delta A,B)$, $(A,B+\delta B)$ and $(A+\delta A,B+\delta B)$. In time t these move to $(X(A,B,t), Y(A,B,t))$, $(X(A+\delta A,B,t), Y(A+\delta A,B,t))$, $(X(A,B+\delta B,t), Y(A,B+\delta B,t))$ and $(X(A+\delta A, B+\delta B,t), Y(A+\delta A,B+\delta B,t))$ (see Figure 3.2.1).

The area of the resulting element is

$$\delta A\delta B(X_A Y_B - X_B Y_A) + \text{higher order terms.}$$

For an incompressible fluid this area is equal to $\delta A\delta B$. Letting δA and δB tend to zero, we then have

$$X_A Y_B - X_B Y_A = 1 . \tag{3.2.5}$$

In some cases it may be convenient to eliminate P from (3.2.3) and (3.2.4). To do this differentiate (3.2.3) with respect to B and (3.2.4) with respect to A.

Then/

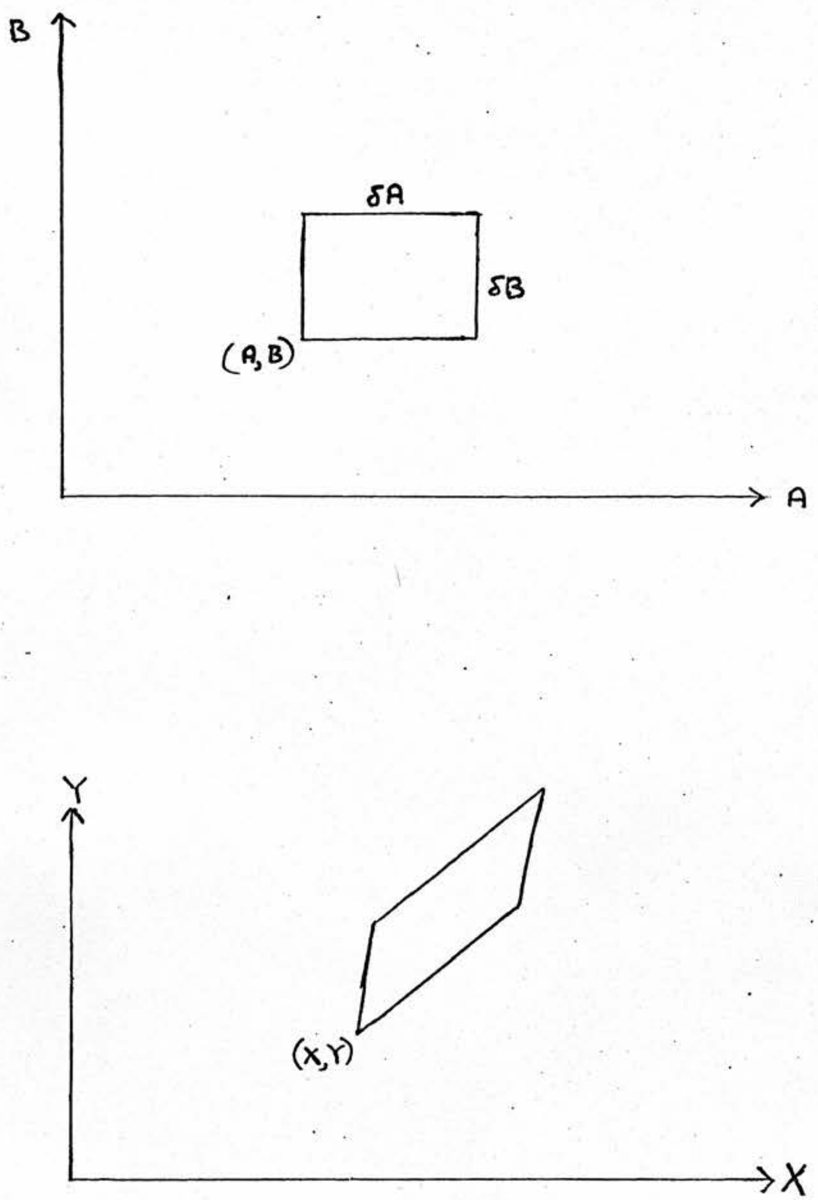


FIG. 3.2.1

Change in a small element of fluid

Then $X_{tt}X_{AB} + X_{Btt}X_A + Y_{tt}Y_{AB} + Y_{Btt}Y_A = -P_{AB}/\rho - g\,Y_{AB}$

and $X_{tt}X_{AB} + X_{Att}X_B + Y_{tt}Y_{AB} + Y_{Att}Y_B = -P_{AB}/\rho - g\,Y_{AB}$.

Subtract to get $X_{Btt}X_A - X_{Att}X_B + Y_{Btt}Y_A - Y_{Att}Y_B = 0$.

This may be written as $\frac{\partial}{\partial t}(X_{Bt}X_A - X_{At}X_B + Y_{Bt}Y_A - Y_{At}Y_B) = 0$.

Integration gives $-X_{Bt}X_A + X_{At}X_B - Y_{Bt}Y_A + Y_{At}Y_B = \Gamma(A,B)$.

It may be shown that the expression on the left-hand side corresponds to the vorticity. Hence, for any problem starting from rest or with zero vorticity this becomes

$$X_{At}X_B - X_{Bt}X_A + Y_{At}Y_B - Y_{Bt}Y_A = 0 \quad . \tag{3.2.6}$$

(The derivation of equations (3.2.1)-(3.2.6) may be found in Stoker (1957) - Chapter 12.)

Now consider the water entry problem - a wedge of angle 2α enters a half-space of water at constant speed V_o (see Figure 3.2.2).

In the absence of gravity and surface tension, or more accurately, on the assumption that their effect is negligible over some period of time, it is apparent that a similarity solution will exist. Now $X/V_o t$ is a dimensionless quantity and must be a function of dimensionless variables. X can only depend on A,B,V_o,t,α and the only dimensionless combinations of A,B,V_o and t are A/V_ot, B/V_ot and $A/B = (A/V_ot)/(B/V_ot)$). Hence it is possible to write

$$X(A,B,t) = V_o t\, x(a,b) \quad \text{and} \quad Y(A,B,t) = V_o t\, y(a,b)$$

where $A = V_o t\, a$ and $B = V_o t\, b$. If the pressure is involved then we may write $P(A,B,t) = \rho V_o^2\, \pi(a,b)$. The equations

of/

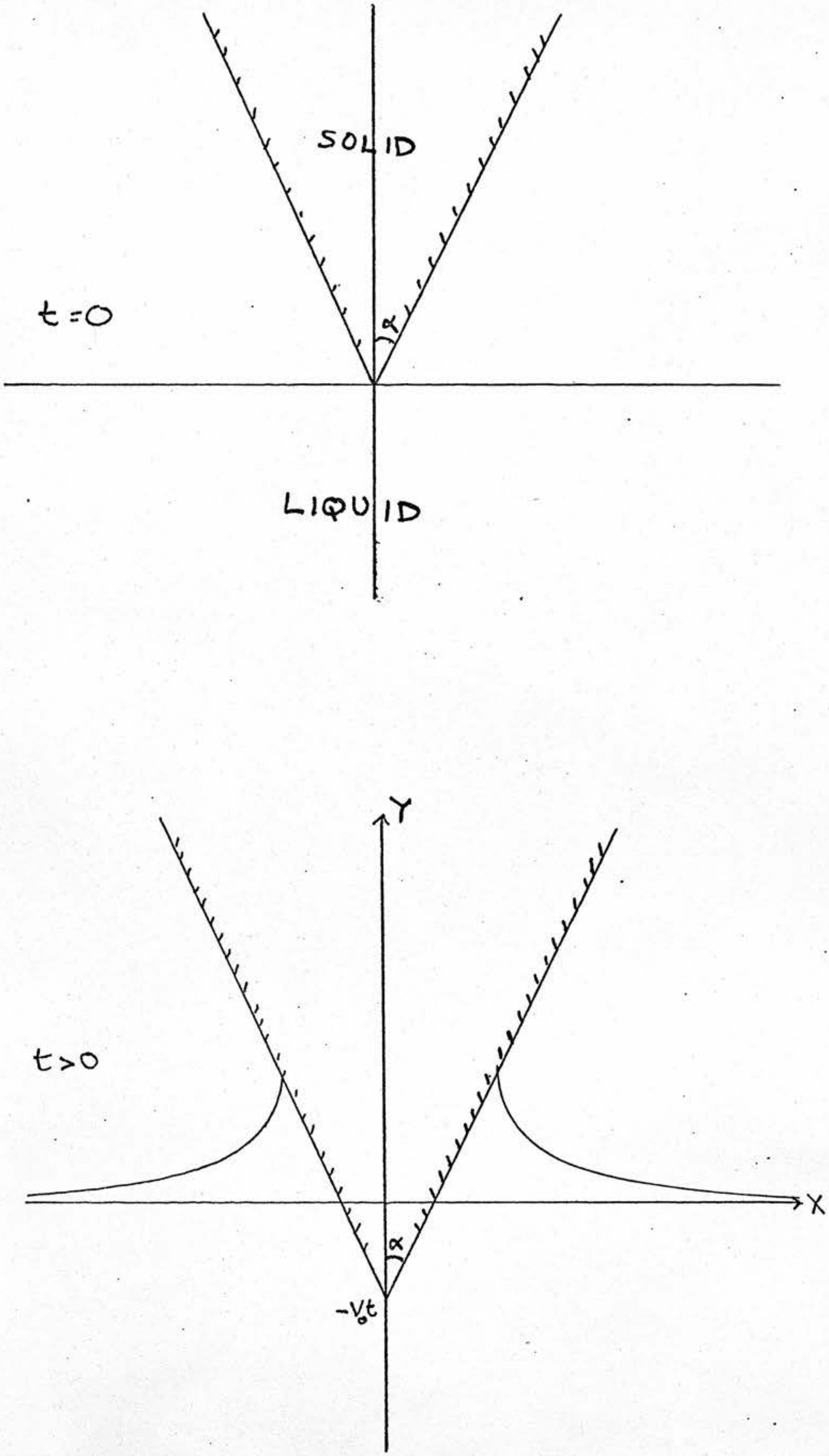


FIG. 3.2.2

Water entry of a wedge

of motion (3.2.3)-(3.2.6) may now be written in terms of these similarity variables.

Since $X(A,B,t) \;=\; V_o t\, x(a,b)$,

$$X_A \;=\; x_a \;,$$

and $$X_{At} \;=\; -\frac{1}{t}\,(ax_{aa} + bx_{ab}) \;.$$

Also $X_t \;=\; V_o(x - ax_a - bx_b)$

and $X_{tt} \;=\; (V_o/t)(a^2x_{aa} + 2ab^2x_{ab} + b\,x_{bb})$.

$X_B, Y_A, Y_B, X_{Bt}, Y_{At}, Y_{Bt}, Y_{tt}$ follow by symmetry.

Finally, since $P(A,B,t) \;=\; \rho V_o^{\,2}\,\pi(a,b)$, we have

$$P_A \;=\; \frac{\rho V_o}{t}\,\pi_a \;.$$

Then (3.2.3) and (3.2.4) become respectively

$$(a^2x_{aa} + 2ab\,x_{ab} + b^2x_{bb})x_a + (a^2y_{aa} + 2ab\,y_{ab} + b^2y_{bb})y_a \;=\; -\pi_a . \quad (3.2.7)$$

$$(a^2x_{aa} + 2ab\,x_{ab} + b^2x_{bb})x_b + (a^2y_{aa} + 2ab\,y_{ab} + b^2y_{bb})y_b \;=\; -\pi_b . \quad (3.2.8)$$

(3.2.6) becomes

$$ax_bx_{aa} + (bx_b - ax_a)x_{ab} - bx_ax_{bb}$$

$$+\, ay_by_{aa} + (by_b - ay_a)y_{ab} - by_ay_{bb} \;=\; 0 \;. \quad (3.2.9)$$

The continuity equation (3.2.5) becomes simply

$$x_ay_b - x_by_a \;=\; 1 \;. \quad (3.2.10)$$

We notice that (3.2.10) is a first order equation whereas (3.2.7)-(3.2.9) are second order equations, a fact which may cause some problems when analysing the equations. However, an alternative to (3.2.10) may be obtained by differentiating (3.2.5) with respect to t which gives

$$X_{At}Y_B + X_AY_{Bt} - X_{Bt}Y_A - X_BY_{At} \;=\; 0 \;,$$

a/

a form of the continuity equation which remains valid even when A and B are not the initial values of X and Y. Introducing the similarity variables into this equation, we have

$$a y_b x_{aa} + (b y_b - a y_a) x_{ab} - b y_a x_{bb}$$
$$-a x_b y_{aa} - (b x_b - a x_a) y_{ab} + b x_a y_{bb} = 0 . \qquad (3.2.11)$$

The problem is now to solve either equations (3.2.7) and (3.2.8) together with one of (3.2.10) or (3.2.11) or equation (3.2.9) again with one of (3.2.10) or (3.2.11), subject to suitable boundary conditions.

(3.3) Conservation of arc-length on the free surface.

Theorem: Suppose we have a flow governed by the equations (3.2.7)-(3.2.10). Then consider a curve composed of fluid particles which initially lies on a straight line through the origin. Then arc-length (measured along the curve) on some segment of this curve is invariant during some period of time if and only if the pressure is constant on that segment during that time.

Proof: Consider a straight line $b = ma$.

Let $\sigma^2 = a^2 + b^2$ so that $a = \sigma/\sqrt{1+m^2}$ and $b = m\sigma/\sqrt{1+m^2}$.

Then on $b = ma$, $x_\sigma = x_a a_\sigma + x_b b_\sigma$

$$\Rightarrow x_\sigma = (x_a + m x_b)/\sqrt{1+m^2} .$$

Hence $x_{\sigma\sigma} = (x_{aa} + 2m x_{ab} + m^2 x_{bb})/(1+m^2)$.

Multiply (3.2.8) by m and add the result to (3.2.7).

Then

$$(a^2 x_{aa} + 2ab x_{ab} + b^2 x_{bb})(x_a + m x_b) + (a^2 y_{aa} + 2ab y_{ab} + b^2 y_{bb})(y_a + m y_b)$$
$$= -(\pi_a + m\pi_b) .$$

On/

On $b = ma$, this becomes

$$x_{\sigma\sigma} x_\sigma + y_{\sigma\sigma} y_\sigma = -\pi_\sigma/\sigma^2$$

$$\Rightarrow \frac{d}{d\sigma}(x_\sigma^2 + y_\sigma^2) = -\frac{2\pi_\sigma}{\sigma^2}$$

$$\Rightarrow \frac{d}{d\sigma}(x_\sigma^2 + y_\sigma^2) = 0 \Leftrightarrow \pi_\sigma = 0$$

$$\Rightarrow x_\sigma^2 + y_\sigma^2 = \text{const.} \Leftrightarrow \pi = \text{const} .$$

The result follows since $x_\sigma^2 + y_\sigma^2 = X_s^2 + Y_s^2$ where $s = V_o t\sigma$. This theorem was first proved by Wagner (1932) and subsequently by Garabedian (1953) and Mackie (1969) although none of these authors used the Lagrangian form of the equations of motion to obtain the result. In the case of the water-entry problem the free surface corresponds to the line $b = 0$. The pressure is zero on the free surface throughout the motion and so the above theorem may be applied. In this case $m = 0$ and so $\sigma = a$. Thus on $b = 0$, $x_a^2 + y_a^2 = \text{const}$. But $x_a = X_A$ and $y_a = Y_A$. Moreover $X = A$ and $Y = 0$ at $t = 0$ on $B = 0$, so that $X_A = 1$ and $Y_A = 0$ at $t = 0$. Hence $x_a^2 + y_a^2 = 1$ on $b = 0$.

(3.4) Boundary conditions for the water-entry problem.

Because of the symmetry of the problem it is only necessary to consider its solution in the quarter-plane $A \geq 0$, $B \leq 0$, or in the similarity variables $a \geq 0$, $b \leq 0$. There are four distinct parts of the boundary to be considered: the free surface AB, BC, CD and the wedge face AD (see Figure 3.4.1).

It is assumed that particles initially on the free surface remain there/

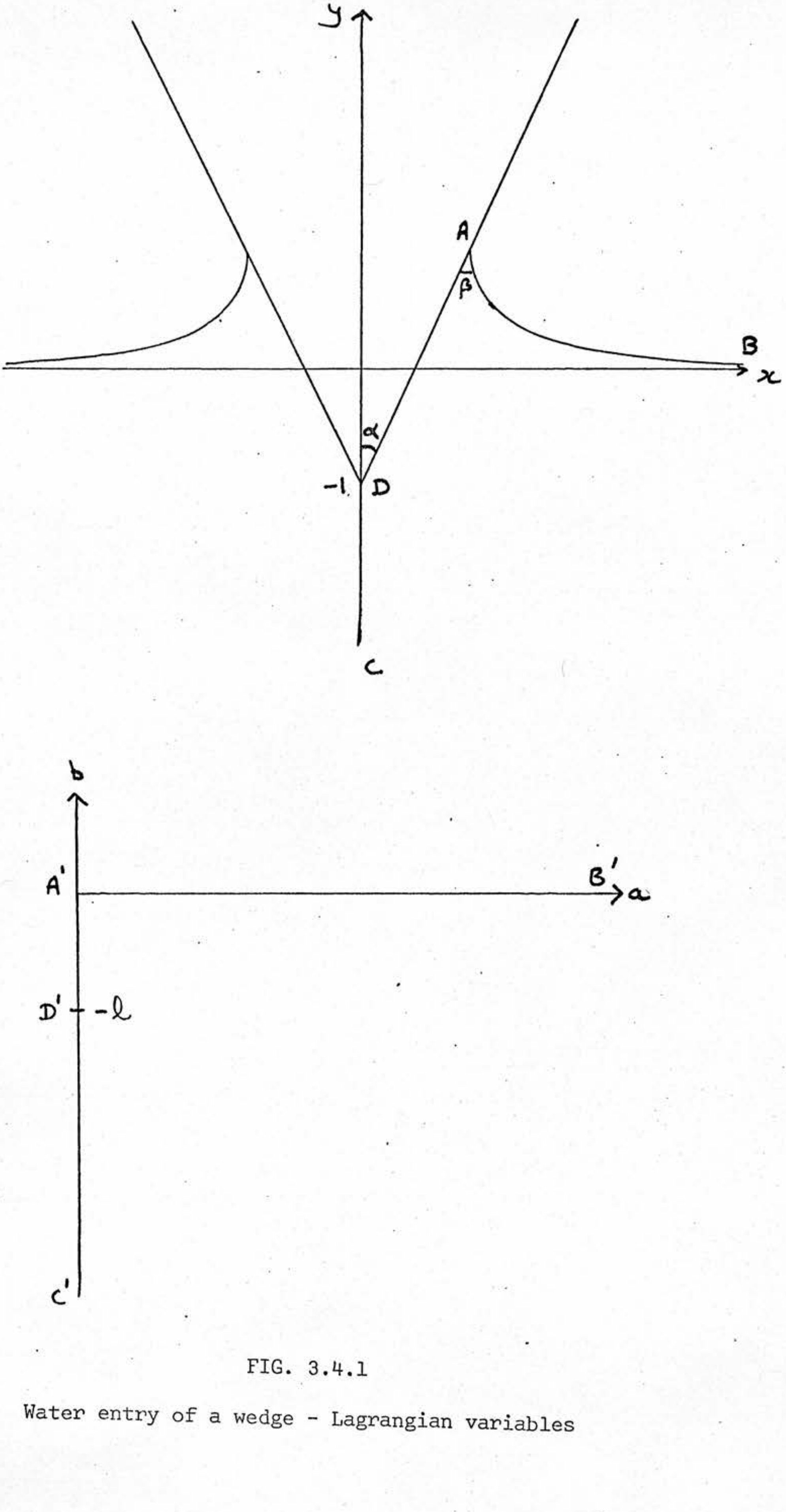


FIG. 3.4.1

Water entry of a wedge - Lagrangian variables there throughout the motion. This implies that the free surface AB corresponds to the line $b = 0$. By symmetry the line ADC consists of the particles for which $a = 0$. However it is not known at this stage which point corresponds to D. It is necessary therefore to introduce an unknown quantity ℓ such that $x(0, -\ell) = 0$ and $y(0, -\ell) = -1$. Because of the similarity ℓ is a dimensionless constant. Consider a particle initially at $(0,B)$ where $B = V_o tb$. Then this particle will reach the tip of the wedge at a time $t = t_B$ where, by similarity, $t_B = B/\mu V_o$ for some constant μ.

Now at $t = t_B$, $X(0,B,t_B) = 0$ and $Y(0,B,t_B) = -V_o t_B$.

Hence $V_o t_B \; x(0, B/V_o t_B) = 0$

$$\Rightarrow x(0,\mu) = 0 .$$

Also $V_o t_B \, y(0, B/V_o t_B) = -V_o t_B$.

$$\Rightarrow y(0,\mu) = -1 .$$

But the quantity $-\ell$ satisfies $x(0,-\ell) = 0$ and $y(0,-\ell) = -1$.

Thus $\mu = -\ell$ and $t_B = -B/\ell V_o$.

In other words, for any particle initially on the Y-axis the constant ℓ gives a measure of the time taken by that particle to reach the tip of the wedge.

Now on $a = 0$, $b \leqslant -\ell$,

$$x(0,b) = 0 \tag{3.4.1}$$

since these particles correspond to the line DC.

Also on $a = 0$, $-\ell \leqslant b \leqslant 0$,

$$x(0,b) = (y(0,b) + 1)\tan\alpha \tag{3.4.2}$$

since these particles correspond to the line AD. On the free surface $b = 0$, the boundary condition is that the pressure

is/

is zero. By the theorem of (3.3) this is equivalent to

$$x_a^2(a,0) + y_a^2(a,0) = 1 . \qquad (3.4.3)$$

At infinity the motion dies away and so we have

$X(A,B,t) = X(A,B,0) = A$ and $Y(A,B,t) = Y(A,B,0) = B$

or $x(a,b) \to a$ and $y(a,b) \to b$ as $a^2 + b^2 \to \infty$. (3.4.4)

(3.5) Treatment of the equations of motion as a first order system.

Consider equations (3.2.9) and (3.2.11).

These are respectively

$$ax_b x_{aa} + (bx_b - ax_a)x_{ab} - bx_a x_{bb} + ay_b y_{aa} + (by_b - ay_a)y_{ab} - by_a y_{bb} = 0$$

$$ay_b x_{aa} + (by_b - ay_a)x_{ab} - by_a x_{bb} - ax_b y_{aa} - (bx_b - ax_a)y_{ab} + bx_a y_{bb} = 0 .$$

To reduce these to a first order system write

$$\begin{aligned} p &= x_a \\ q &= x_b \\ r &= y_a \\ s &= y_b . \end{aligned}$$

The first of the equations becomes

$$aqp_a + bqp_b - apq_a - bpq_b + asr_a + bsr_b - ars_a - brs_b = 0 \qquad (3.5.1)$$

and the second becomes

$$asp_a + bsp_b - arq_a - brq_b - aqr_a - bqr_b + aps_a + bps_b = 0 . \qquad (3.5.2)$$

There are also the consistency relations

$$\begin{aligned} p_b &= q_a \\ r_b &= s_a . \end{aligned}$$

These four equations can be written in the form

$$A\,\underline{u}_a + B\,\underline{u}_b = 0 \qquad (3.5.3)$$

where/

where

$$A = \begin{bmatrix} aq & -ap & as & -ar \\ as & -ar & -aq & +ap \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} \quad B = \begin{bmatrix} bq & -bp & bs & -br \\ bs & -br & -bq & bp \\ -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \end{bmatrix} \quad \underline{u} = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} .$$

The characteristics for this system can be found as follows.

Let λ be an eigenvalue such that $\det(A - \lambda B) = 0$.

The quantity $d\underline{u}$ is given by

$$d\underline{u} = da\, \underline{u}_a + db\, \underline{u}_b . \qquad (3.5.4)$$

Multiply (3.5.3) by da and (3.5.4) by A.

Then $\quad da\, A\, \underline{u}_a + da\, B\underline{u}_b = 0$

$\quad\quad da\, A\, \underline{u}_a + db\, A\underline{u}_b = Ad\underline{u}$.

Subtract to obtain $(Adb - Bda)\underline{u}_b = Ad\underline{u}$.

Let $\underline{v}$ be a left eigenvector such that $\underline{v}^T(A - \lambda B) = 0$.

Then $\underline{v}^T(Adb - Bda)\underline{u}_b = \underline{v}^T A\, d\underline{u}$.

Hence if $\lambda = da/db$, $\underline{v}^T (Adb - Bda) = 0$ and

$$\underline{v}^T A\, d\underline{u} = 0 . \qquad (3.5.5)$$

The characteristics are given by $da/db = \lambda$ and on these curves the relation $\underline{v}^T A\, d\underline{u} = 0$ holds.

Now

$$\det(A-\lambda B) = \det \begin{bmatrix} aq-\lambda bq & -ap+\lambda bp & as-\lambda bs & -ar+\lambda br \\ as-\lambda bs & -ar+\lambda br & -aq+\lambda bq & ap-\lambda bp \\ \lambda & 1 & \cdot & \cdot \\ \cdot & \cdot & \lambda & 1 \end{bmatrix}$$

$$= (a-\lambda b)^2 \det \begin{bmatrix} q & -p & s & -r \\ s & -r & -q & p \\ \lambda & 1 & \cdot & \cdot \\ \cdot & \cdot & \lambda & 1 \end{bmatrix}$$

$$= (a-\lambda b)^2 \left\{\lambda^2(p^2+ r^2) + 2\lambda(pq+rs) + (q^2+ s^2)\right\} .$$

Thus/

Thus $\det(A-\lambda B) = 0 \Rightarrow \lambda = a/b$ (twice)

$$\text{and } \lambda = -\frac{(pq+rs) \pm i}{p^2+r^2} \quad \text{using (3.2.10).}$$

The real characteristics are given by

$$\frac{da}{db} = \frac{a}{b}$$

$\Rightarrow$ $b = ma$ for constant m.

The real characteristics are thus rays through the origin, the same curves which occur in Theorem (3.3).

Let $\underline{v}$ be a left eigenvector for $\lambda = a/b$ where

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} .$$

Then $\underline{v}^T A = (a/b)\, \underline{v}^T B$.

Hence

$$b(v_1 aq + v_2 as) = a(v_1 bq + v_2 bs - v_3 b)$$

$$b(-v_1 ap - v_2 ar + v_3) = a(-v_1 bp - v_2 br)$$

$$b(v_1 as - v_2 aq) = a(v_1 bs - v_2 bq - v_4 b)$$

$$b(-v_1 ar + v_2 ap + v_4) = a(-v_1 br + v_2 bp) .$$

There are two independent eigenvectors for $\lambda = a/b$ and these may be written as $\{1 \ 0 \ 0 \ 0\}^T$ and $\{0 \ 1 \ 0 \ 0\}^T$.

In either case the relation $\underline{v}^T A\, d\underline{u} = 0$ holds.

If $\underline{v}^T = \{1 \ 0 \ 0 \ 0\}$, $\{aq \ \ -ap \ \ as \ \ -ar\}\{dp \ \ dq \ \ dr \ \ ds\}^T = 0$.

$$\Rightarrow q\, dp - p\, dq + s\, dr - r\, ds = 0 . \qquad (3.5.6)$$

If $\underline{v}^T = \{0 \ 1 \ 0 \ 0\}$, $\{as \ \ -ar \ \ -aq \ \ ap\}\{dp \ \ dq \ \ dr \ \ ds\}^T = 0$

$$\Rightarrow s\, dp - r\, dq - q\, dr + p\, ds = 0$$

$$\Rightarrow ps - qr = \text{const} = 1 \text{ . (from (3.2.10))} \qquad (3.5.7)$$

We know already that $ps - qr = 1$ everywhere - some supposedly 'new' information has arisen simply because information was destroyed/

destroyed in setting up the equations (3.5.3). However (3.5.6) is more important. It gives a new relation which holds on all rays through the origin including the lines $a = 0$ and $b = 0$. That it holds on the boundary can easily be checked by putting a and b equal to zero in turn in the equations of motion (3.5.3).

(3.6) The behaviour of certain flow quantities on the free surface.

Theorem: Consider a flow governed by the equations (3.5.3). Suppose there is a free surface which initially lay along the line $B = 0$ and thus is subsequently asymptotic to the line $Y = 0$, assuming that the fluid remains at rest at infinity. Then, for all particles which remain on the free surface throughout the motion, the quantities p, q, r, s can be related to the gradient of the free surface as follows. If the gradient of the free surface is $\tan\phi$, then

$$\begin{aligned} p &= \cos\phi \\ q &= -\sin\phi + 2\phi\cos\phi \\ r &= \sin\phi \\ s &= \cos\phi + 2\phi\sin\phi\ . \end{aligned}$$

(See Figure 3.6.1 for notation).

Proof: The equations (3.5.3) hold. On $b = 0$, the first of these reduces to

$$q\, p_a - p\, q_a + s\, r_a - r\, s_a = 0 \qquad (3.6.1)$$

and the second to

$$p\, s - q\, r = 1 \quad \text{(using (3.2.10))}. \qquad (3.6.2)$$

Since/

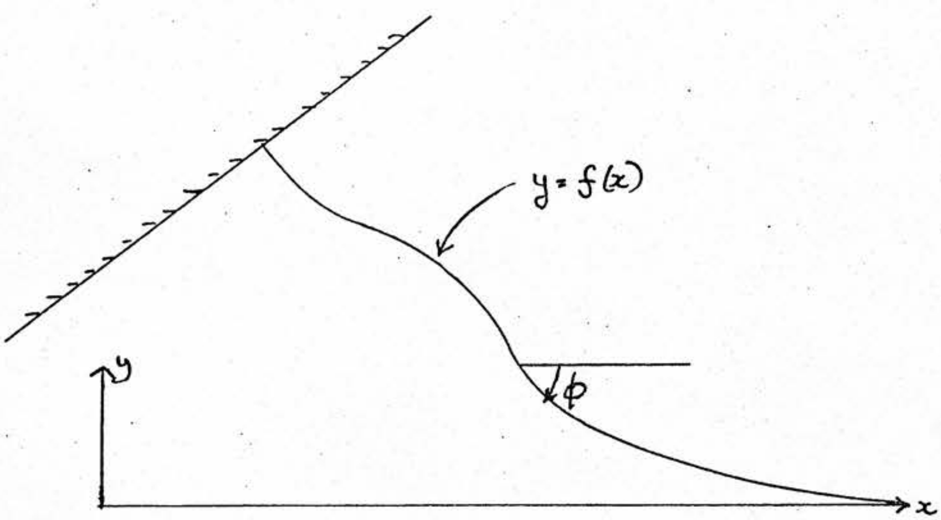


FIG. 3.6.1

A free surface curve

Since the pressure on the free surface is constant and since the free surface initially formed part of a ray through the origin, theorem (3.3) can be applied, and so

$$p^2 + r^2 = 1 \tag{3.6.3}$$

for those particles which remain on the free surface throughout the motion. If the equation of the free surface is $y = f(x)$, or $y(a,0) = f\{x(a,0)\}$, then differentiation gives $p(a,0) = f'(x)\, r(a,0)$. But ϕ is defined so that $\tan\phi = f'(x)$ and thus

$$\tan\phi = \frac{r(a,0)}{p(a,0)} .$$

Now, from (3.6.3), $p^2 + r^2 = 1$ and so

$$p = \pm\cos\phi \quad \text{and} \quad r = \pm\sin\phi .$$

Since $r(a,0) = \lim\limits_{h\to 0} \dfrac{y(a+h,0) - y(a,0)}{h}$

and it is clear from Figure 3.6.1 that this limit is positive when ϕ is positive and negative when ϕ is negative, we have

$$r = \sin\phi \tag{3.6.4}$$

and
$$p = \cos\phi . \tag{3.6.5}$$

This satisfies $p^2 + r^2 = 1$ identically and so we now consider the remaining equations (3.6.2) and (3.6.1). Substitute for p and r from (3.6.4) and (3.6.5) so that (3.6.2) becomes

$$s\cos\phi - q\sin\phi = 1 , \tag{3.6.6}$$

and (3.6.1) becomes

$$q\sin\phi + \cos\phi\,\frac{dq}{d\phi} = -\sin\phi\,\frac{ds}{d\phi} + s\cos\phi . \tag{3.6.7}$$

Differentiation of (3.6.6) yields

$$q\cos\phi + \sin\phi\,\frac{dq}{d\phi} = \cos\phi\,\frac{ds}{d\phi} - s\sin\phi . \tag{3.6.8}$$

Eliminate/

Eliminate $dq/d\phi$ from (3.6.7) and (3.6.8) and substitute for q from (3.6.6). Then

$$\frac{ds}{d\phi} = \frac{s\cos\phi}{\sin\phi} - \frac{\cos 2\phi}{\sin\phi} \ .$$

Hence $s = 2\phi\sin\phi + \cos\phi + A\sin\phi$ (3.6.9)

for some constant A.

Also $\dfrac{dq}{d\phi} = -s + 2\cos\phi$

$\Rightarrow$ $q = A\cos\phi - \sin\phi + 2\phi\cos\phi + B$ (3.6.10)

for constant B.

But we have to satisfy

$$\frac{ds}{d\phi} = 2\sin\phi - q$$

and this is possible only if $B = 0$.

Hence $q = A\cos\phi - \sin\phi + 2\phi\cos\phi$. (3.6.11)

When $\phi = 0$, $s = 1$ and $q = A$.

But the conditions of the theorem require that the fluid is at rest at infinity and so at infinity $p = s = 1$ and $q = r = 0$.

Thus $A = 0$, and gathering together the results from (3.6.4), (3.6.5), (3.6.9) and (3.6.11)

$$\begin{aligned} p &= \cos\phi \\ q &= -\sin\phi + 2\phi\cos\phi \\ r &= \sin\phi \\ s &= \cos\phi + 2\phi\sin\phi \ . \end{aligned}$$

(3.7) Relations on the boundary.

For the water entry problem, the boundary conditions were found in section (3.4) to be:

on $b = 0$, $x_a{}^2 + y_a{}^2 = 1$,

on $a = 0$, $-\ell \leqslant b \leqslant 0$, $x = (y+1)\tan\alpha$,

on $a = 0$, $b \leqslant -\ell$, $x = 0$,

and/

and for large $a^2 + b^2$, $x \to a$ and $y \to b$.

In terms of p,q,r,s these become:

on $b = 0$, $p^2 + r^2 = 1$ (3.7.1)

on $a = 0$, $-\ell < b < 0$, $q = s \tan\alpha$ (3.7.2)

on $a = 0$, $b < -\ell$ $q = 0$ (3.7.3)

and for large $a^2 + b^2$, $p \to 1$, $q \to 0$, $r \to 0$, $s \to 1$. (3.7.4)

Now each of the equations (3.7.1), (3.7.2) and (3.7.3) may in turn be used in conjunction with (3.5.6) and (3.5.7) to say something more about the behaviour of p,q, r and s.

Consider first the line $b = 0$ which corresponds to the free surface. There we have

$$p^2 + r^2 = 1$$
$$ps - qr = 1$$
$$q\,dp - p\,dq + s\,dr - r\,ds = 0 .$$

It follows immediately from Theorem (3.6) that

$$\left.\begin{aligned} p &= \cos\phi \\ q &= -\sin\phi + 2\phi\cos\phi \\ r &= \sin\phi \\ s &= \cos\phi + 2\phi\sin\phi \end{aligned}\right\} \qquad (3.7.5)$$

where the gradient of the free surface is $\tan\phi$.

Note that the assumption that the free surface is convex towards the fluid implies that the angle ϕ is a monotonic increasing function of a .

Since $r(a,0) = f'(x)\,p(a,0)$,

$$r_a = f''(x)\,p^2 + f'(x)p_a$$

$$\Rightarrow \quad f''(x) = \frac{pr_a - rp_a}{p^3}$$

$$\Rightarrow \quad f''(x) = \sec^3\phi\ \phi_a \qquad \text{(using (3.7.5))}.$$

Clearly $f''(x) > 0$ when $-\pi/2 < \phi < 0$ and $f''(x) < 0$

when/

when $\phi < -\pi/2$ provided $\phi_a > 0$. The result follows.

Next consider the line $a = 0$. If $-\ell < b < 0$, we have

$$q = s \tan \alpha \tag{3.7.6}$$

$$ps - qr = 1 \tag{3.7.7}$$

$$q\,dp - p\,dq + s\,dr - r\,ds = 0 . \tag{3.7.8}$$

From (3.7.6) $dq = ds \tan \alpha$. Substitute for q and dq in (3.7.7) and (3.7.8) to get

$$ps - rs \tan \alpha = 1 \tag{3.7.9}$$

$$s \tan \alpha\, dp - p \tan \alpha\, ds + s\,dr - r\,ds = 0 . \tag{3.7.10}$$

Equation (3.7.10) may be integrated to give

$$p \tan \alpha + r = Ks \sec^2 \alpha . \tag{3.7.11}$$

Solve for p and r from (3.7.9) and (3.7.11).

Then
$$\begin{aligned} p &= K s \tan \alpha + \frac{\cos^2\alpha}{s} \\ q &= s \tan \alpha \\ r &= Ks - \frac{\sin \alpha \cos \alpha}{s} \end{aligned} \tag{3.7.12}$$

On $a = 0$, $b < -\ell$, we have

$$q = 0 \tag{3.7.13}$$

$$ps - qr = 1 \tag{3.7.14}$$

$$q\,dp - p\,dq + s\,dr - r\,ds = 0 . \tag{3.7.15}$$

From (3.7.13) $q = dq = 0$.

Hence, from (3.7.14) $p s = 1$

Also from (3.7.15), $r = C s$ for some constant C.

But from (3.7.4) $r \to 0$ and $s \to 1$ as $b \to -\infty$.

Hence $C = 0$ and $r = 0$ for all $b < -\ell$.

Then on $a = 0$, $b < -\ell$, we have $ps = 1$, $q = r = 0$. (3.7.16)

To sum up, the conditions on the various parts of the boundary are as follows:

On/

On the free surface, given by $b = 0$,

$$p = \cos\phi$$

$$q = -\sin\phi + 2\phi\cos\phi$$

$$r = \sin\phi$$

$$s = \cos\phi + 2\phi\sin\phi \,.$$

On $a = 0$, $-\ell < b < 0$, $p = K\,s\tan\alpha + \dfrac{\cos^2\alpha}{s}$

$$q = s\tan\alpha$$

$$r = K\,s - \frac{\sin\alpha\cos\alpha}{s} \,.$$

On $a = 0$, $b < -\ell$, $p\,s = 1$ and $q = r = 0$.

If r and p are both continuous at the point $a = b = 0$, then we can identify the constant K. For $\phi = -(\pi/2-(\beta-\alpha))$ at the origin and so

$$r = \sin\phi = -\cos(\beta-\alpha)$$

$$p = \cos\phi = \sin(\beta-\alpha) \,.$$

But on $a = 0$, $-\ell < b < 0$, $r = K\,s - \sin\alpha\cos\alpha/s$

$$p = K\,s\tan\alpha + \cos^2\alpha/s \,.$$

Equating these values of r and p at the origin and solving for s and K leads to $s = \cos\alpha/\sin\beta$ and

$$K = -\sin\beta\cos\beta.$$

However, there is no guarantee that either p or r is continuous at the origin.

(3.8) The contact angle.

Considerable interest has been shown in the contact angle β which the wedge makes with the free surface. First of all, we notice that, on dimensional grounds, β is a constant in time and depends/

depends only on the wedge angle α. As has already been mentioned in Chapter 2, Dobrovol'skaya (1969), Garabedian (1965) and Mackie (1969) have shown between them that $0 < \beta < \pi/4$, on the assumption that the free surface is convex to the fluid. In Chapter 4 we shall consider the contact angle again from the point of view of a local analysis of the flow near the contact point, but in the meantime we tackle this question from the point of view of the Lagrangian variables.

Consider lines in the a-b plane for which a is constant. These correspond to lines which were initially vertical in the fluid and in this context it may be helpful to think of a and b as A and B at fixed time $t = 1$ and with $V_o = 1$. Under the transformation $x = x(a,b)$, $y = y(a,b)$ these lines are mapped into curves in the x-y plane. Clearly the line $a = 0$ is mapped into the line segments $x = 0$, $y \leqslant -1$ and $x = (y+1)\tan\alpha, -1 \leqslant y \leqslant k\cos\alpha - 1$, where k is an unknown constant and in fact kV_o is the velocity of the front of the fluid up the wedge relative to the wedge. Each of the remaining curves has one end on the free surface while the other end tends to its fixed initial value since $x \to a$ as $b \to -\infty$. Consider a sequence of 'normal line elements' which are defined as line elements on the free surface whose slope is equal to that of the curves a = constant at that point. Let $\tan\psi$ be the slope of one of these elements at such a point (see Figure 3.8.1).

Now $\dfrac{s(a,0)}{q(a,0)} = \lim\limits_{h\to o} \dfrac{y(a,h) - y(a,0)}{x(a,h) - x(a,0)}$

$\Rightarrow$ $q/s = -\cot\lambda$

$\Rightarrow$ $q/s = \tan\psi$ where $\psi = \pi/2 + \lambda$ (see Figure 3.8.1) .

But we know from Theorem 3.6 that, on the free surface,

q/

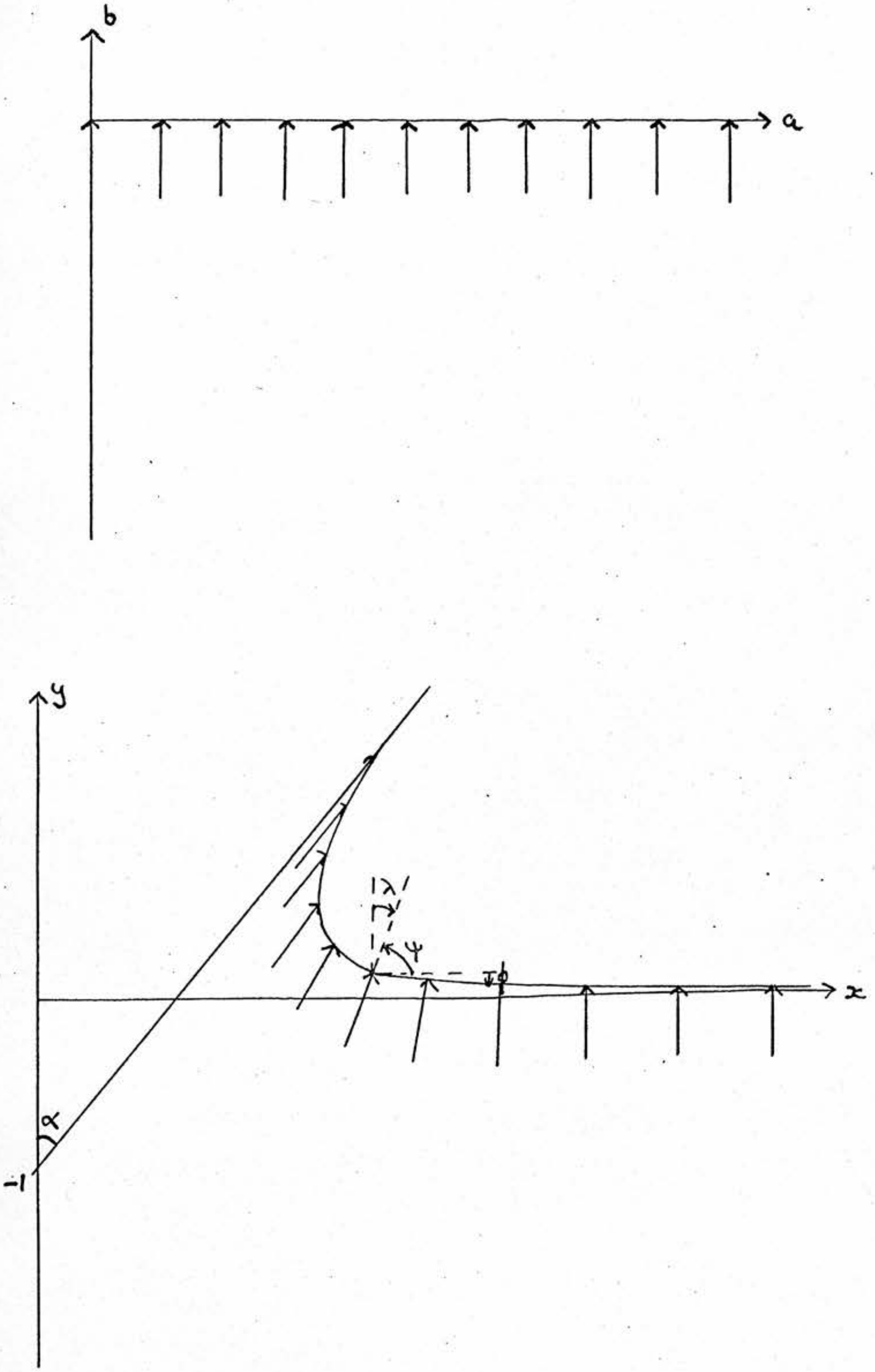


FIG. 3.8.1

Normal line elements

$$q = -\sin\phi + 2\phi\cos\phi \quad\text{and}\quad s = \cos\phi + 2\phi\sin\phi\ .$$

Hence $$\tan\psi = \frac{1+2\phi\tan\phi}{2\phi - \tan\phi}\ . \tag{3.8.1}$$

This is an exact result relating the slope of the normal line elements to the slope of the free surface at the point where they meet the free surface, and furnishes some interesting results concerning the curves given by a = constant. We notice that the slope of the normal line elements is vertical when $\psi = \pi/2$ and this occurs when $2\phi = \tan\phi$. This equation has two roots in the range $-\pi \leqslant \phi \leqslant 0$, namely $\phi = 0$ and $\phi = \phi_1 \approx -67^{\circ}$. For $\phi_1 < \phi < 0$, $2\phi - \tan\phi < 0$, $\tan\psi < 0$ and $\psi > \pi/2$. For $\phi_o < \phi < \phi_1$, $2\phi - \tan\phi > 0$, $\tan\psi > 0$ and $\psi < \pi/2$, where $\phi_o = -(\pi/2 - (\beta-\alpha))$. Now since arc-length is conserved on the free surface, each point on the free surface moves to the right of its initial position (as well as upwards), and since the lines of constant a are fixed at infinity, it follows that each line for which $\phi > \phi_1$, and thus $\psi > \pi/2$, has at least one inflexion (see Figure 3.8.2).

It is not certain that the angle ϕ_1 is attained for all values of α, but it certainly occurs for large enough α, since whenever $\alpha \geqslant \pi/4$, $\alpha > \beta$ (since $\beta < \pi/4$) and the free surface has a vertical tangent so that the maximum value of ϕ is greater than $\pi/2$ and so greater than ϕ_1. Then, on the assumption that the free surface is convex to the fluid, we can conclude that ϕ_1 is attained at exactly one point on the free surface.

From the formula (3.8.1) it might be possible to say something about the contact angle β. For, as $a \to 0$, $\phi \to -(\pi/2 - (\beta-\alpha))$ and, provided the slope of the normal line elements/

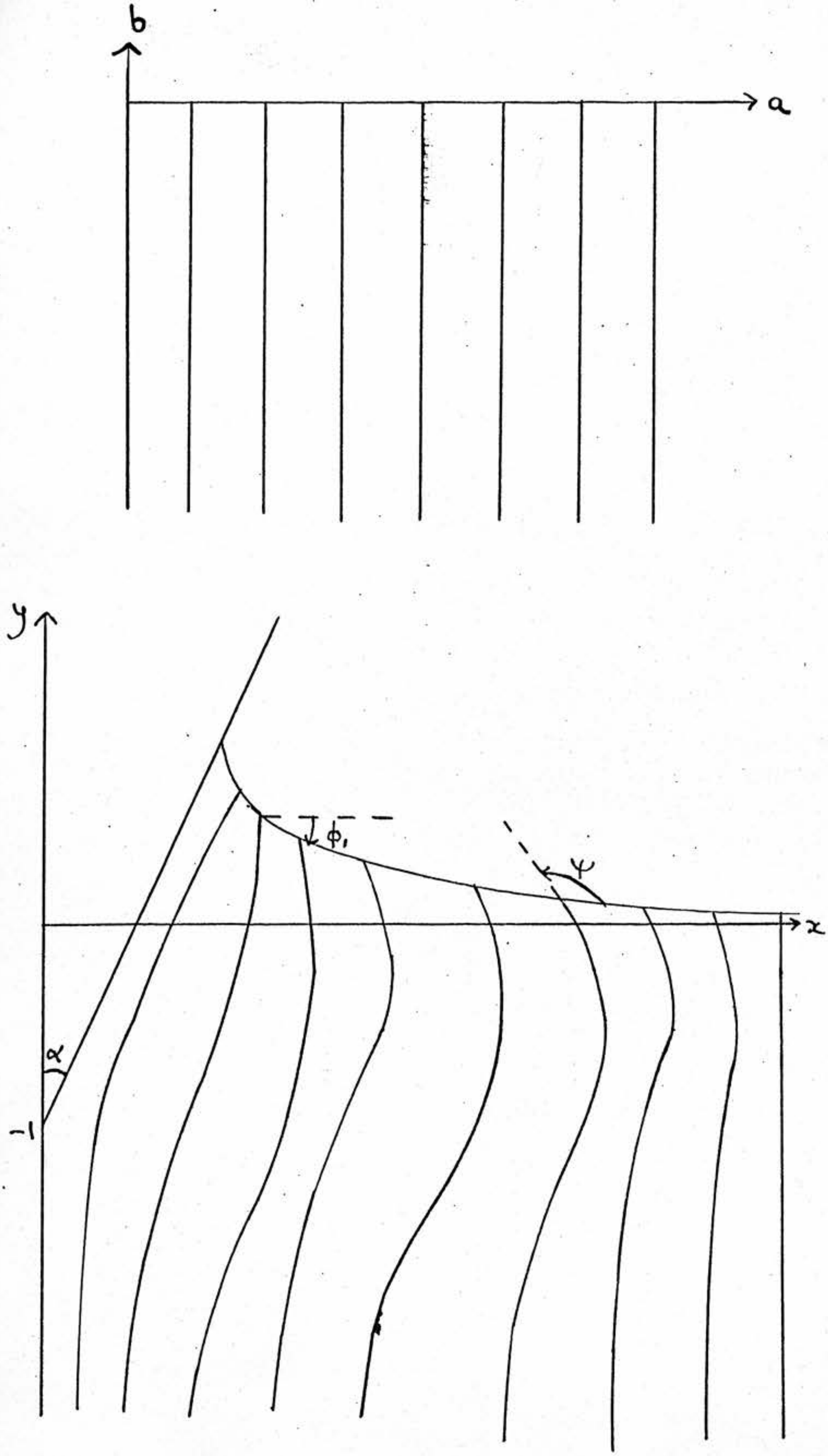


FIG. 3.8.2

Curves for which a is constant

elements is continuous, then $\psi \to (\pi/2 - \alpha)$. Substitution of these values of ϕ and ψ in (3.8.1) yields

$$\cot\alpha = \frac{1 + 2(\pi/2-(\beta-\alpha))\cot(\beta-\alpha)}{\cot(\beta-\alpha) - 2(\pi/2 -(\beta-\alpha))}$$

$$\Rightarrow \quad \cot\beta + 2\beta = \pi + 2\alpha \quad . \qquad (3.8.2)$$

It is a simple matter to solve this equation for β, given a value of α between zero and $\pi/2$.

It is clear from Figure 3.8.3 that for various values of α between zero and $\pi/2$ there is a unique value for β (at any rate in the range $(0,\pi/2)$) which is a solution of (3.8.2). Moreover, for $\alpha = 0$, we can have $\beta = \pi/2$ which is the correct solution for a wedge of zero thickness, although this is not the limit as α tends to zero of the value of β for non-zero α. This demonstrates the singular nature of the problem as α tends to zero.

Some simple calculations yield the following results:

α(degrees)	β(degrees)
0·0	23·2183
1·0	22·7788
5·0	21·2208
10·0	19·6153
20·0	17·1464
30·0	15·3031
40·0	13·8559
50·0	12·6808
60·0	11·7032
70·0	10·8744
80·0	10·1612
90·0	9·5401

These results are represented graphically in Figure 3.8.4.

These results are interesting from various points of view. In terms of the most straightforward interpretation they would seem to furnish a simple formula for calculating the contact angle for any given wedge angle. They do not contradict the fact that β has been shown, by a number of writers independently, to lie in the/

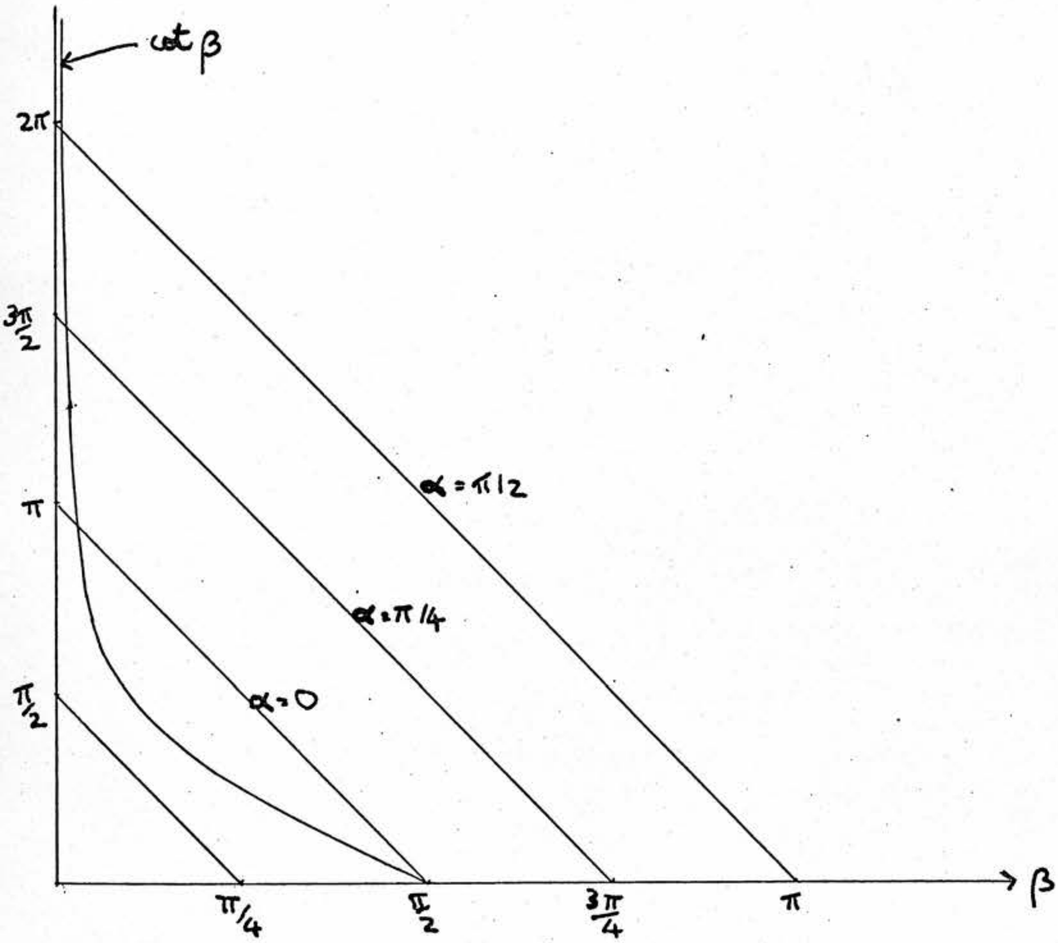


FIG. 3.8.3

Graph of cot β and lines $\pi+2\alpha-2\beta$ against β for various values of α

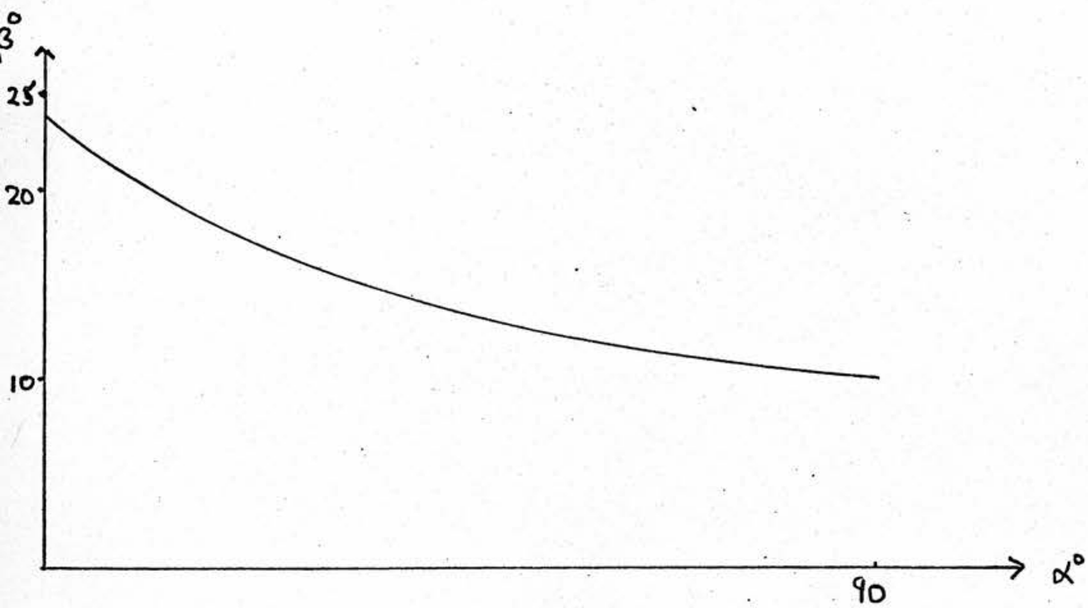


FIG. 3.8.4

Graph of the curve $\cot\beta+2\beta = \pi+2\alpha$

the range $0 < \beta < \pi/4$. In fact they support the stronger conjecture that $0 < \beta < \pi/6$ which is the case if the curvature of the free surface at the contact point is zero (see Mackie (1969)). However there are two points where contradictions are in evidence. In the first place Mackie has shown that, as a consequence of the convexity of the free surface and conservation of arc-length along the free surface, α and β must satisfy the inequality $\alpha+\beta \lesssim \pi/2$. The above results do not agree with this when α is large - in fact they contradict it for $\alpha > \alpha_o$ where $\tan \alpha_o = 4\alpha_o$ or $\alpha_o \approx 79^o$. The other context in which a contradiction arises is in comparison with the numerical results of Dobrovol'skaya (1969). She finds that, as α increases from 0^o to 90^o, β decreases from about 18^o to 0^o, but the shape of the function $\beta(\alpha)$ is very similar to the one shown in Figure 3.8.4.

The simplest explanation of these discrepancies is that the assumption of continuity of s/q is incorrect and that the slope of the normal line elements is discontinuous at the contact point. This is a feasible solution to the problem since one might expect complicated behaviour of the various flow quantities in the neighbourhood of that point. If such a discontinuity of slope exists, the question arises of why it does so. It may be due to the presence of the corner at the tip of the wedge, or to the self-similarity inherent in the problem, or to the presence of the free surface. To check on the first of these, a series of calculations was done for flow past wedges of different angles without either self-similarity or a free surface being taken into account. In no case/

case was there any obvious evidence of a discontinuity of the type that was being sought (see Appendix I for details). The other two possibilities remain.

It is possible to consider also the nature of the postulated discontinuity in the slope of the normal line-elements. As $a \to 0$, it is clear that $\phi \to \phi_o = -\{\pi/2 - (\beta-\alpha)\}$ and that there is no discontinuity in ϕ. Then the relation (3.8.1) defines a limiting angle ψ_o. If ψ_o is not equal to $(\pi/2-\alpha)$ then we can ask whether it is less than or greater than $(\pi/2-\alpha)$. ψ_o is defined by (3.8.1) as

$$\tan\psi_o = \frac{1 + 2\phi_o \tan\phi_o}{2\phi_o - \tan\phi_o}$$

$$\Rightarrow \cot(\psi_o - \phi_o) = 2\phi_o$$

$$\Rightarrow \pi+2\alpha - 2\beta = \tan(\psi_o + \alpha-\beta) \quad \text{since} \quad \phi_o = -\{\pi/2-(\beta-\alpha)\} .$$

If $\psi_o > \pi/2-\alpha$, $\pi + 2\alpha-2\beta = \tan(\psi_o + \alpha-\beta)$

$$\Rightarrow \pi + 2\alpha - 2\beta > \cot\beta .$$

If $\psi_o < \pi/2-\alpha$, $\pi + 2\alpha - 2\beta < \cot\beta$.

In the latter case the curve $2\beta + \cot\beta = \pi + 2\alpha$ acts as an upper bound for $\beta(\alpha)$, and both Dobrovol'skaya's numerical results and the inequality $\alpha+\beta \leqslant \pi/2$ could be reconciled with the present theory. In order to reach such a conslusion we should need to show that the normal line elements 'overshoot' their expected value (fig. 3.8.5), but so far nothing positive has been achieved in this direction.

There are, however, other possible explanations of the contradictions which have arisen. It must be noted that no existence and uniqueness theorems have been proved for the solution of/

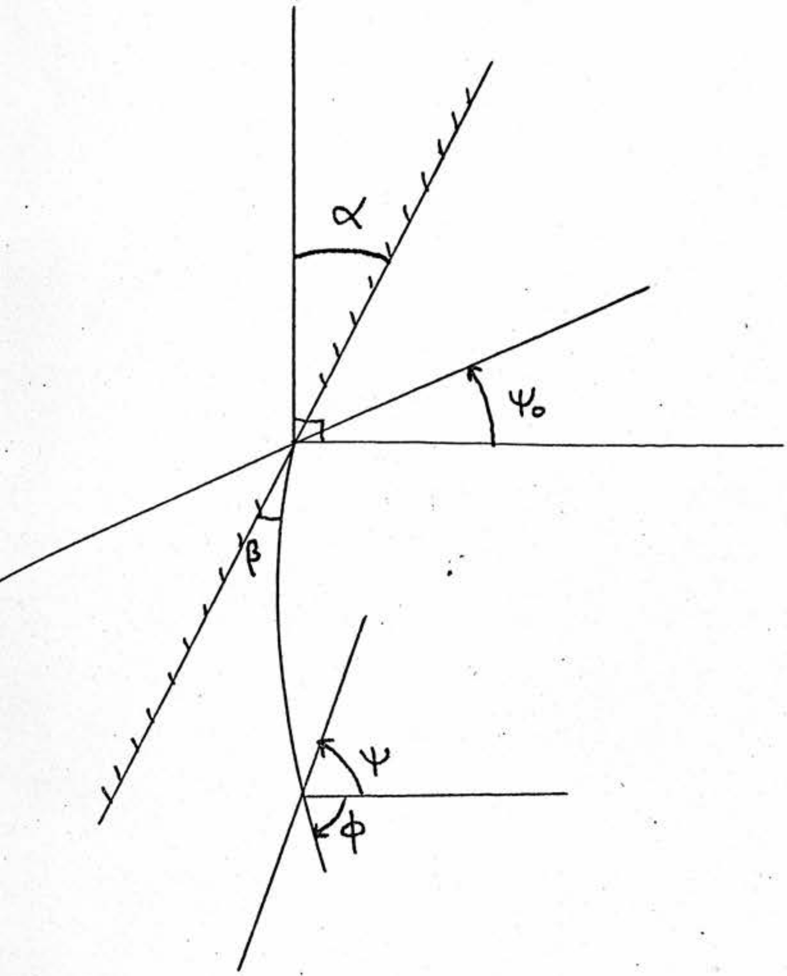


FIG. 3.8.5

Possible configuration for the normal line elements
near the contact point

of the water-entry problem, either in its Eulerian or Lagrangian formulation. Since this is the case, any results produced about the solution to this problem have to be examined very carefully. For example, consider Dobrovol'skaya's numerical results. If we had existence and uniqueness theorems then Dobrovol'skaya's integral equation would give the unique solution, as much accuracy as desired could be obtained by further calculations, and even the Lagrangian displacements computed, since this is essentially a matter of calculating the particle paths for a known potential function. In the absence of sufficient theoretical support, we see that Dobrovol'skaya's calculations were only performed with reasonable accuracy for wedge angles up to 9°. For wedge angles of 30° and 60° they are only approximate and it appears to be merely fortuitous that she has managed to satisfy the inequality $\alpha+\beta \lesssim \pi/2$ as there are no calculations for large wedge angles. Nevertheless this is a relatively minor point and there is no good reason to suppose that her figures are in general suspect, provided the validity of the basic integral equation is accepted. There is also the question of the inequality $\alpha+\beta \lesssim \pi/2$. The validity of this depends on two things - the convexity of the free surface and the conservation of arc length along the free surface. The first of these conditions is equivalent to the assertion that the pressure is non-negative on the wedge face (see section 2.4) - this seems a reasonable assumption but it cannot be verified *a priori*. Certainly the problem for the velocity potential ϕ as posed by $\nabla^2\phi = 0$ together with boundary conditions (2.4.2)-(2.4.5) in no way prejudges the issue of convexity of the free surface. Then there/

there is the matter of conservation of arc-length. The theorem actually holds only for particles which are on the free surface throughout the motion, but is generally assumed to hold over the whole length of the free surface and in particular takes for granted that all the particles initially on the free surface constitute the whole of the free surface at later times. This implies that the particle which was initially at the origin is subsequently exactly that particle which is situated at the separation point. Again this seems an attractive proposition but cannot apparently be justified rigourously. The boundary value problem for $\phi(x,y)$ is not obviously dependent on any such hypothesis, but of course the formulation of boundary conditions for the Lagrangian displacements $x(a,b)$ and $y(a,b)$ does depend on making some such assumption. It is necessary to consider these questions carefully because the complex variable methods employed by Mackie and Dobrovol'skaya depend fundamentally on the convexity of the free surface - the mapping by the Wagner function is invalid unless this condition is satisfied. Dobrovol'skaya has also used the arc-conservation property over the whole length of the free surface as a check on the accuracy of her numerical results and Mackie has used the same condition to produce the inequality $\alpha+\beta \lesssim \pi/2$.

What has emerged then is that in order to make any progress certain apparently reasonable assumptions must be made. In the Eulerian problem it is very difficult to proceed without making the assumption of convexity of the free surface from the beginning, whereas in the Lagrangian problem one needs to assume at least qualitative knowledge of what happens to particles on the boundary of the fluid.

(3.9)/

(3.9) The auxiliary functions $A(\rho,\theta)$ and $B(\rho,\theta)$.

Consider the equations (3.2.9) and (3.2.10). These are

$$ax_b x_{aa} + (bx_b - ax_a)x_{ab} - bx_a x_{bb}$$
$$+ ay_b y_{aa} + (by_b - ay_a)y_{ab} - by_a y_{bb} = 0 ,$$
$$x_a y_b - x_b y_a = 1 .$$

It has been shown in section (3.5) that an equivalent system of first order partial differential equations has a family of real characteristics which are rays through the origin. It might be advantageous then to rewrite the equations in polar coordinates so that the characteristics simply become lines of constant θ .

Let $a = \rho\cos\theta$ and $b = \rho\sin\theta$.

Then $x_a = x_\rho \cos\theta - \dfrac{\sin\theta}{\rho} x_\theta$

and $x_b = x_\rho \sin\theta + \dfrac{\cos\theta}{\rho} x_\theta$.

Hence (3.2.9) becomes

$$x_\theta x_{\rho\rho} - x_\rho x_{\rho\theta} + \frac{x_\rho x_\theta}{\rho} + y_\theta y_{\rho\rho} - y_\rho y_{\rho\theta} + \frac{y_\rho y_\theta}{\rho} = 0 \qquad (3.9.1)$$

while (3.2.10) becomes

$$x_\rho y_\theta - x_\theta y_\rho = \rho . \qquad (3.9.2)$$

These have to be solved in the quarter plane $-\pi/2 \leqslant \theta \leqslant 0$ and boundary conditions follow immediately from (3.4.1)-(3.4.4). These are now:

$$x = 0 \quad \text{on} \quad \theta = -\pi/2,\ \rho \geqslant \ell \qquad (3.9.3)$$
$$x = (y+1)\tan\alpha \quad \text{on} \quad \theta = -\pi/2,\ 0 \leqslant \rho \leqslant \ell \qquad (3.9.4)$$
$$x_\rho^{\,2} + y_\rho^{\,2} = 1 \quad \text{on} \quad \theta = 0,\ \rho \geqslant 0 \qquad (3.9.5)$$
$$x \to \rho\cos\theta \ \text{and}\ y \to \rho\sin\theta \ \text{as}\ \rho \to \infty . \qquad (3.9.6)$$

Now/

Now equation (3.9.1) may be rewritten as

$$\frac{\partial}{\partial\theta}(\rho(x_\rho{}^2 + y_\rho{}^2)) = \frac{\partial}{\partial\rho}(\rho(x_\rho x_\theta + y_\rho y_\theta)) .$$

We can introduce an auxiliary function $A(\rho,\theta)$ defined by

$$A_\rho = \rho(x_\rho{}^2 + y_\rho{}^2) \tag{3.9.7}$$

$$A_\theta = \rho(x_\rho x_\theta + y_\rho y_\theta) . \tag{3.9.8}$$

If we write $\zeta = x+iy$ then (3.9.2), (3.9.7) and (3.9.8) can all be combined in the statements

$$A_\rho = \rho\, \zeta_\rho \overline{\zeta}_\rho \tag{3.9.9}$$

$$A_\theta = \rho\, \overline{\zeta}_\rho \zeta_\theta - i\,\rho^2. \tag{3.9.10}$$

Division of these gives $\zeta_\theta = \dfrac{A_\theta + i\rho^2}{A_\rho}\zeta_\rho$ (3.9.11)

and differentiating with respect to ρ we have

$$\zeta_{\rho\theta} = \frac{\partial}{\partial\rho}\left(\frac{A_\theta + i\rho^2}{A_\rho}\right)\zeta_\rho + \left(\frac{A_\theta + i\rho^2}{A_\rho}\right)\zeta_{\rho\rho} . \tag{3.9.12}$$

Now equations (3.9.1) and (3.9.2) may be combined as

$$(\rho\overline{\zeta}_\rho)_\rho \zeta_\theta - \rho\zeta_\rho \overline{\zeta}_{\rho\theta} = 2i\rho \tag{3.9.13}$$

and substituting for $\zeta_{\rho\theta}$ from (3.9.12) leads to

$$\rho\zeta_\rho\overline{\zeta}_{\rho\rho}\left(\frac{2i\rho^2}{A_\rho}\right) + \frac{A_\theta + i\rho^2}{\rho} - A_\rho \frac{\partial}{\partial\rho}\left(\frac{A_\theta - i\rho^2}{A_\rho}\right) = 2i\rho . \tag{3.9.14}$$

The imaginary part of (3.9.14) is an identity but the real part gives

$$-x_\rho y_{\rho\rho} + y_\rho x_{\rho\rho} = \frac{A_\rho}{2\rho^3}\left\{\frac{A_\theta}{\rho} - A_\rho \left(\frac{A_\theta}{\rho A\rho}\right)_\rho\right\} . \tag{3.9.15}$$

Also from (3.9.9) we have

$$x_\rho x_{\rho\rho} + y_\rho y_{\rho\rho} = \tfrac{1}{2}\left\{\frac{A_{\rho\rho}}{\rho} - \frac{A_\rho}{\rho^2}\right\} . \tag{3.9.16}$$

If/

If these last two equations are solved for $x_{\rho\rho}$ and $y_{\rho\rho}$,

then $$\frac{\zeta_{\rho\rho}}{\zeta_\rho} = \tfrac{1}{2}\left\{ \frac{A_{\rho\rho}}{A_\rho} - \frac{1}{\rho} + \frac{i}{\rho} A_\rho \left(\frac{A_\theta}{\rho A_\rho}\right)_\rho \right\} . \qquad (3.9.17)$$

Let $$B_\rho = \frac{A_\rho}{\rho} \frac{\partial}{\partial\rho}\left(\frac{A_\theta}{\rho A_\rho}\right) . \qquad (3.9.18)$$

Then integration of (3.9.17) gives

$$\log \zeta_\rho = \tfrac{1}{2} \log \frac{A_\rho}{\rho} + \frac{i}{2} B + i\, H(\theta) .$$

$H(\theta)$ is real since (3.9.9) implies $|\zeta_\rho| = \sqrt{\frac{A_\rho}{\rho}}$.

Now as $\rho \to \infty$, $x \sim \rho\cos\theta$ and $y \sim \rho \sin\theta$ so that $A_\rho \sim \rho$ and $B = O(\rho^{-3})$. Since $\zeta \sim \rho e^{i\theta}$, we must have $H(\theta) \equiv \theta$, and hence

$$\zeta_\rho = \sqrt{\frac{A_\rho}{\rho}}\; e^{i(B/2+\theta)} . \qquad (3.9.19)$$

From (3.9.11) $$\zeta_\theta = \frac{A_\theta + i\rho^2}{A_\rho} \sqrt{\frac{A_\rho}{\rho}}\; e^{i(B/2+\theta)} . \qquad (3.9.20)$$

Differentiate (3.9.19) with respect to θ and (3.9.20) with respect to ρ and equate $\zeta_{\rho\theta}$ and $\zeta_{\theta\rho}$.

Then $$B_\theta + 2 = \frac{3\rho}{A_\rho} - \rho^2 \frac{A_{\rho\rho}}{A_\rho^{\,2}} + \frac{A_\theta}{\rho} \frac{\partial}{\partial\rho}\left(\frac{A_\theta}{\rho A_\rho}\right) . \qquad (3.9.21)$$

From (3.9.18) and (3.9.21), if $B_{\rho\theta} = B_{\theta\rho}$, we have the following third order equation for $A(\rho,\theta)$.

$$\begin{aligned}&\rho^2 A_\rho (A_\theta^{\,2} A_{\rho\rho\rho} - 2A_\rho A_\theta A_{\rho\rho\theta} + A_\rho^{\,2} A_{\rho\theta\theta}) - \rho^2 (2A_\theta^{\,2} A_{\rho\rho}^{\,2} + 4A_\rho A_\theta A_{\rho\theta} A_{\rho\rho} - A_\rho^{\,2} A_{\rho\theta}^{\,2} \\ &\quad - A_\rho^{\,2} A_{\rho\rho} A_{\theta\theta}) - \rho A\rho (3A_\theta^{\,2} A_{\rho\rho} + 4A_\rho A_\theta A_{\rho\theta} - A_\rho^{\,2} A_{\theta\theta}) - 3A_\rho^{\,2} A_\theta^{\,2} \\ &\quad + \rho^4 (\rho^2 A_\rho A_{\rho\rho\rho} - 2\rho^2 A_{\rho\rho}^{\,2} + 5\rho A_\rho A_{\rho\rho} - 3A_\rho^{\,2}) = 0 . \qquad (3.9.22)\end{aligned}$$

From (3.9.19) and (3.9.20) we can write $x_\rho, x_\theta, y_\rho, y_\theta$ in terms of A and B as follows

$$\begin{aligned} x_\rho &= \sqrt{\frac{A_\rho}{\rho}} \cos(B/2 + \theta) \\ x_\theta &= \frac{A_\theta}{A_\rho} \sqrt{\frac{A_\rho}{\rho}} \cos(B/2 + \theta) - \frac{\rho^2}{A_\rho} \sqrt{\frac{A_\rho}{\rho}} \sin(B/2 + \theta) \end{aligned} \qquad (3.9.23)$$

y/

$$y_\rho = \sqrt{\frac{A_\rho}{\rho}} \sin(B/2 + \theta)$$

$$y_\theta = \frac{A_\theta}{A_\rho} \sqrt{\frac{A_\rho}{\rho}} \sin(B/2 + \theta) + \frac{\rho^2}{A_\rho} \sqrt{\frac{A_\rho}{\rho}} \cos(B/2 + \theta) \;.$$

Ideally, one might attempt to solve equation (3.9.22) for $A(\rho,\theta)$, find $B(\rho,\theta)$ from (3.9.18) and (3.9.21) and finally $x(\rho,\theta)$ and $y(\rho,\theta)$ from (3.9.23). The problem has been reduced to that of finding a single function, albeit the partial differential equation for that function is formidable.

Now although the function $A(\rho,\theta)$ has been described as an auxiliary function, it is in fact closely related to the pressure in the fluid. The pressure $\pi(\rho,\theta)$ may be found from equations (3.2.7) and (3.2.8) which are

$$(a^2x_{aa} + 2ab\, x_{ab} + b^2x_{bb})x_a + (a^2y_{aa} + 2ab\, y_{ab} + b^2y_{bb})y_a = -\pi_a$$

$$(a^2x_{aa} + 2ab\, x_{ab} + b^2x_{bb})x_b + (a^2y_{aa} + 2ab\, y_{ab} + b^2y_{bb})y_b = -\pi_b \;.$$

Writing these in terms of the polar coordinates ρ,θ we see that

$$-\pi_\rho = \rho^2(x_\rho x_{\rho\rho} + y_\rho y_{\rho\rho})$$

$$-\pi_\theta = \rho^2(x_{\rho\rho}x_\theta + y_{\rho\rho}y_\theta) \;.$$

Hence $-\pi_\rho = \dfrac{\rho^2}{2} \dfrac{\partial}{\partial\rho}(x_\rho^{\,2} + y_\rho^{\,2})$

and $-\pi_\theta = \rho^2\left\{\dfrac{\partial}{\partial\rho}(x_\rho x_\theta + y_\rho y_\theta) - \tfrac{1}{2}\dfrac{\partial}{\partial\theta}(x_p^{\,2} + y_\rho^{\,2})\right\} \;.$

Using the definition of A from (3.9.7) and (3.9.8) we have

$$-\pi_\rho = \frac{\rho^2}{2} \frac{\partial}{\partial\rho}\left(\frac{A_\rho}{\rho}\right) \qquad (3.9.24)$$

$$-\pi_\theta = \rho^2\left\{\frac{\partial}{\partial\rho}\left(\frac{A_\theta}{\rho}\right) - \frac{1}{2}\frac{\partial}{\partial\theta}\left(\frac{A_\rho}{\rho}\right)\right\} \;. \qquad (3.9.25)$$

If we integrate (3.9.24) by parts we have the simple result

$$-\pi = \frac{\rho A_\rho}{2} - A + \lambda(\theta) \qquad \text{for some function } \lambda \;.$$

As $\rho \to \infty$, $A \sim \rho^2/2$ and $\pi \sim 0$.

Hence/

Hence $\lambda(\theta) \equiv 0$ and the pressure is given by

$$\pi = A - \rho A_\rho/2 . \tag{3.9.26}$$

It is possible to formulate boundary conditions for the functions A and B from the known boundary conditions for x and y given by (3.9.3)-(3.9.6).

Since $A_\rho = \rho(x_\rho^2 + y_\rho^2)$ (3.9.5) implies that $A_\rho = \rho$ on $\theta=0$.

Then $B_\rho = (A_\rho/\rho) \frac{\partial}{\partial\rho} \left(\frac{A_\theta}{\rho A_\rho}\right)$

$$\Rightarrow \quad B_\rho = \frac{\partial}{\partial\rho}\left(\frac{A_\theta}{\rho^2}\right) \qquad \text{on } \theta = 0$$

$$\Rightarrow \quad B = A_\theta/\rho^2 , \qquad \text{on } \theta = 0 .$$

But $A_\theta = \rho^2(x_\rho x_\theta + y_\rho y_\theta)$ by definition and so

$$A_\theta = \rho^2(pq + rs) \qquad \text{on } \theta = 0.$$

We know from (3.7.4) that on $\theta = 0$,

$$\begin{aligned} p &= \cos\phi \\ q &= -\sin\phi + 2\phi\cos\phi \\ r &= \sin\phi \\ s &= \cos\phi + 2\phi\sin\phi , \end{aligned}$$

and so $A_\theta = 2\phi\,\rho^2$ on $\theta = 0$.

Hence on $\theta = 0$,

$$\begin{aligned} A &= \rho^2/2 \\ B &= 2\phi \end{aligned} \tag{3.9.27}$$

On $\theta = -\pi/2$, $0 < \rho < \ell$, $x_\rho = y_\rho \tan\alpha$ from (3.9.4) .

Using this in conjunction with (3.9.23) we find that

$$\cos(B/2 - \pi/2) = \tan\alpha \sin(B/2 - \pi/2).$$

The simplest solution is $B = -2\alpha$.

Since $B = -2\alpha$, $B_\rho = 0$ and so $(A_\rho/\rho)\, \partial/\partial\rho\, (A_\theta/\rho A_\rho) = 0$.

Clearly A_ρ is not zero over the whole interval, since this would give $x_\rho = y_\rho = 0$, or x and y both constant, which is untrue. The alternative is

$\partial/$

$$\frac{\partial}{\partial\rho}\left(\frac{A_\theta}{\rho A_\rho}\right) = 0 .$$

$\Rightarrow$ $A_\theta + K\rho A_\rho = 0$ for some constant K.

Hence on $\theta = -\pi/2$, $0 < \rho < \ell$, $A_\theta + K\rho A_\rho = 0$ (3.9.28)

$B = -2\alpha$.

On $\theta = -\pi/2$, $\rho > \ell$, $x_\rho = 0$ from (3.9.3).

Using (3.9.23) we have $\cos(B/2 - \pi/2) = 0$.

The simplest solution is $B = 0$.

Since $B = 0$, $B_\rho = 0$ and $(A_\rho/\rho)\, \partial/\partial\rho\, (A_\theta/\rho A_\rho) = 0$.

Again it is clear that $A_\rho \neq 0$ and so $A_\theta/\rho A_\rho$ is constant.

Now as $\rho \to \infty$, $A \to \rho^2/2$ since $x \to \rho \cos\theta$ and $y \to \rho \sin\theta$.

Since $A_\theta/\rho A_\rho$ = constant for $\rho > \ell$ the constant is zero and $A_\theta = 0$.

Hence on $\theta = -\pi/2$, $\rho > \ell$, $A_\theta = 0$

$B = 0$. (3.9.29)

It is possible to be fairly precise about the behaviour of A and B at infinity. It has been stated that the boundary conditions at infinity are $x \to \rho\cos\theta$, $y \to \rho\sin\theta$. However, as all the quantities x,y and ρ become infinite this is not really a sufficiently precise description. It would seem reasonable on physical grounds to assume that the flow at infinity is like that due to a dipole. The tip of the wedge produces an effect like that of a source moving into the fluid and since the velocity potential is zero on the free surface at sufficiently large distances from the origin, we can postulate an image sink so that the source and sink together appear like a dipole at infinity. For the Eulerian problem it can be shown that the behaviour at infinity must be like that due to a dipole if the pressure condition is to be satisfied on the free surface (see Mackie (1969)). This result cannot be applied blindly to the Lagrangian/

Lagrangian problem since it is not necessarily the case that the two ways of formulating the problem are equivalent. Nevertheless, let us make the reasonable assumption that the behaviour at infinity is like that due to a dipole. In this case it can be shown (see Appendix II) that the correct asymptotic behaviour for x and y is

$$\begin{aligned} x &\sim \rho\cos\theta - \frac{m\sin 2\theta}{3\rho^2} \\ y &\sim \rho\sin\theta + \frac{m\cos 2\theta}{3\rho^2} \quad . \end{aligned} \tag{3.9.30}$$

The constant m is related to the strength of the postulated dipole.

Then
$$\begin{aligned} x_\rho &\sim \cos\theta + \frac{2m\sin 2\theta}{3\rho^3} \\ x_\theta &\sim -\rho\sin\theta - \frac{2m\cos 2\theta}{3\rho^3} \\ y_\rho &\sim \sin\theta - \frac{2m\cos 2\theta}{3\rho^3} \\ y_\theta &\sim \rho\cos\theta - \frac{2m\sin 2\theta}{3\rho^2} \quad . \end{aligned}$$

Since $A_\rho = \rho(x_\rho{}^2 + y_\rho{}^2)$ and $A_\theta = \rho(x_\rho x_\theta + y_\rho y_\theta)$ by definition,

$$A \sim \frac{\rho^2}{2} - \frac{4m\sin\theta}{3\rho} + \text{constant.}$$

We shall take the constant to be zero to be consistent with the statement that $\pi = A - \rho A_\rho/2$.

Hence
$$A \sim \frac{\rho^2}{2} - \frac{4m\sin\theta}{3\rho} \quad , \tag{3.9.31}$$

Then
$$A_\rho \sim \rho + \frac{4m\sin\theta}{3\rho^2}$$

and
$$A_\theta \sim -\frac{4m\cos\theta}{3\rho} \quad .$$

Since/

Since $\quad B_\rho = \frac{A_\rho}{\rho} \frac{\partial}{\partial\rho}\left(\frac{A_\theta}{\rho A_\rho}\right)$

and $\quad B_\theta + 2 = \frac{3\rho}{A_\rho} - \rho^2 \frac{A_{\rho\rho}}{A_\rho^{\,2}} + \frac{A_\theta}{\rho} \frac{\partial}{\partial\rho}\left(\frac{A_\theta}{\rho A_\rho}\right)$,

$$B \sim -\frac{4m \cos\theta}{3\rho^3} + \text{constant.}$$

Again, if we are to satisfy $B = A_\theta/\rho^2$ on $\theta = 0$, the constant is zero.

Hence $$B \sim -\frac{4m \cos\theta}{3\rho^3} \quad . \tag{3.9.32}$$

(3.10) Convected rays and circles.

It has been shown (see 3.5) that the partial differential equations under consideration possess a real family of characteristics, namely the rays θ = constant. These rays are mapped into curves, which may be called convected rays (a definition due to Mackie (1968)), in the x-y plane by the mapping $x = x(\rho,\theta)$, $y = y(\rho,\theta)$. In the case of one particular member of this family, the ray $\theta = 0$, a relationship has been found between the slope of the curve $x = x(\rho,0)$, $y = y(\rho,0)$ and the slope of the corresponding normal line element, and this relationship is given by (3.8.1). Let us attempt to generalize this concept to the entire family of characteristics and to their orthogonal trajectories which are circles centre the origin. The rays and circles have images in the x-y plane, convected rays and circles, and a pattern such as that which is shown in Figure 3.10.1 would be expected. Consider a point in the x-y plane where a convected ray and a convected circle intersect. Then/

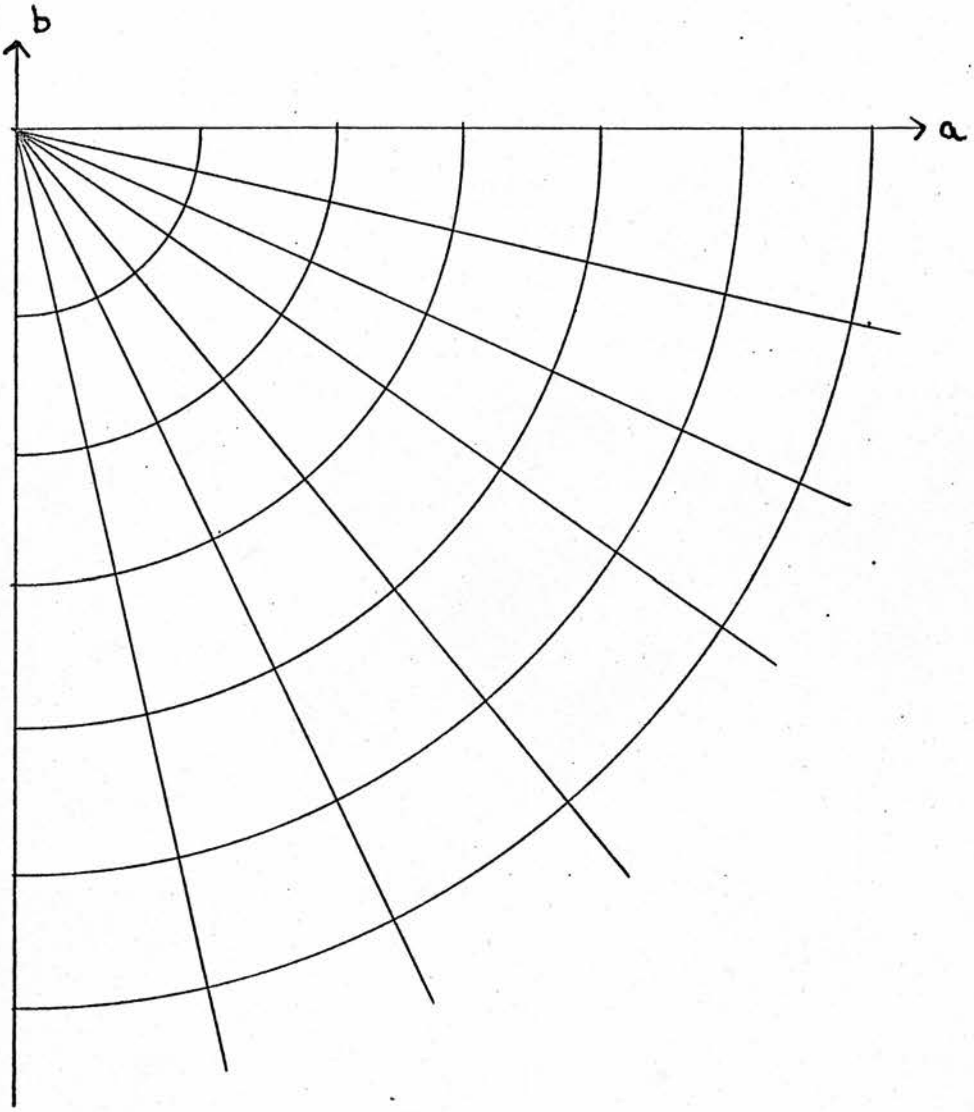


FIG. 3.10.1

Rays and circles (see also following page.)

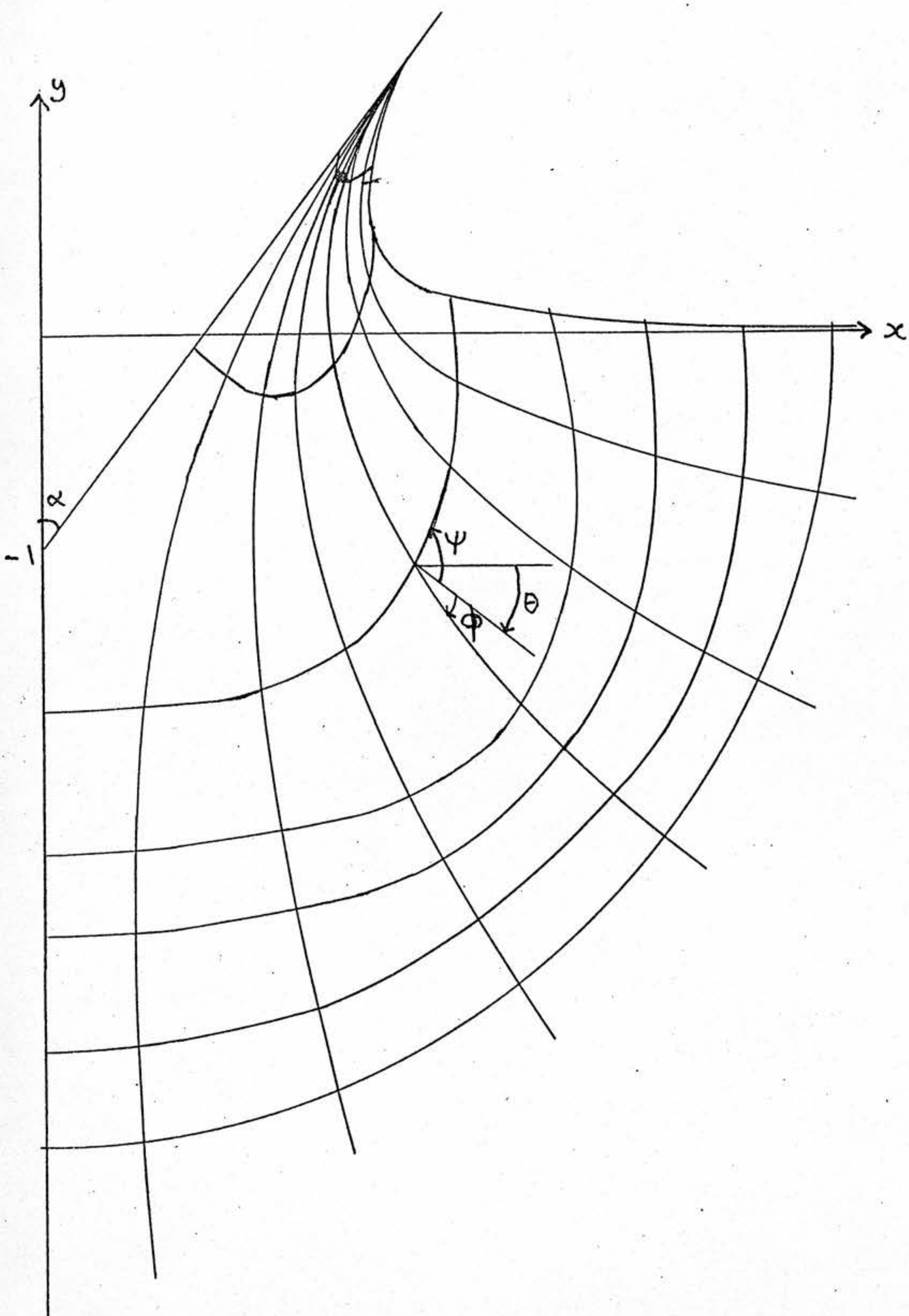


FIG. 3.10.1

Convected rays and circles (see also previous page)

Then there is an angle θ which characterises the initial direction of the convected ray. Let the convected ray make an angle ϕ with this initial direction and let the convected circle make an angle ψ with the same direction. The angles ϕ and ψ are both measured from the initial line in an anti-clockwise direction, so that in Figure 3.10.2 ψ is positive and ϕ is negative. The angle between the convected ray and the convected cirele at any point is then $(\psi - \phi)$.

$$\text{Now} \quad \frac{y_\rho}{x_\rho} = \lim_{h\to o} \frac{y(\rho+h,\theta) - y(\rho,\theta)}{x(\rho+h,\theta) - x(\rho,\theta)}$$

$$\Rightarrow \quad \frac{y_\rho}{x_\rho} = \tan(\theta + \phi) \quad . \qquad (3.10.1)$$

$$\text{Similarly} \quad \frac{y_\rho}{x_\rho} = \lim_{h\to o} \frac{y(\rho,\theta+h) - y(\rho,\theta)}{x(\rho,\theta+h) - x(\rho,\theta)}$$

$$\Rightarrow \quad \frac{y_\rho}{x_\rho} = \tan(\theta+\psi) \quad . \qquad (3.10.2)$$

These are evident from Figure 3.10.2, since θ is constant along the convected ray and ρ is constant along the convected circle. From (3.9.23) we have immediately that

$$\tan(\theta + \phi) = \tan(\theta + B/2)$$

$$\Rightarrow B = 2\phi \quad . \qquad (3.10.3)$$

Since by definition, $A_\theta = \rho(x_\rho x_\theta + y_\rho y_\theta)$, (3.10.2) and (3.10.1) give

$$A_\theta = \rho\, x_\rho x_\theta (1 + \tan(\theta+\phi)\tan(\theta+\psi)) \quad .$$

But the continuity equation is $x_\rho y_\theta - x_\theta y_\rho = \rho$ (from(3.9.2)) and substitution from (3.10.1) and (3.10.2) gives

$$x_\rho x_\theta = \rho/(\tan(\theta+\psi) - \tan(\theta+\phi)) \quad .$$

$$\text{Hence} \quad A_\theta = \rho^2 \cot(\psi-\phi) \quad . \qquad (3.10.4)$$

The/

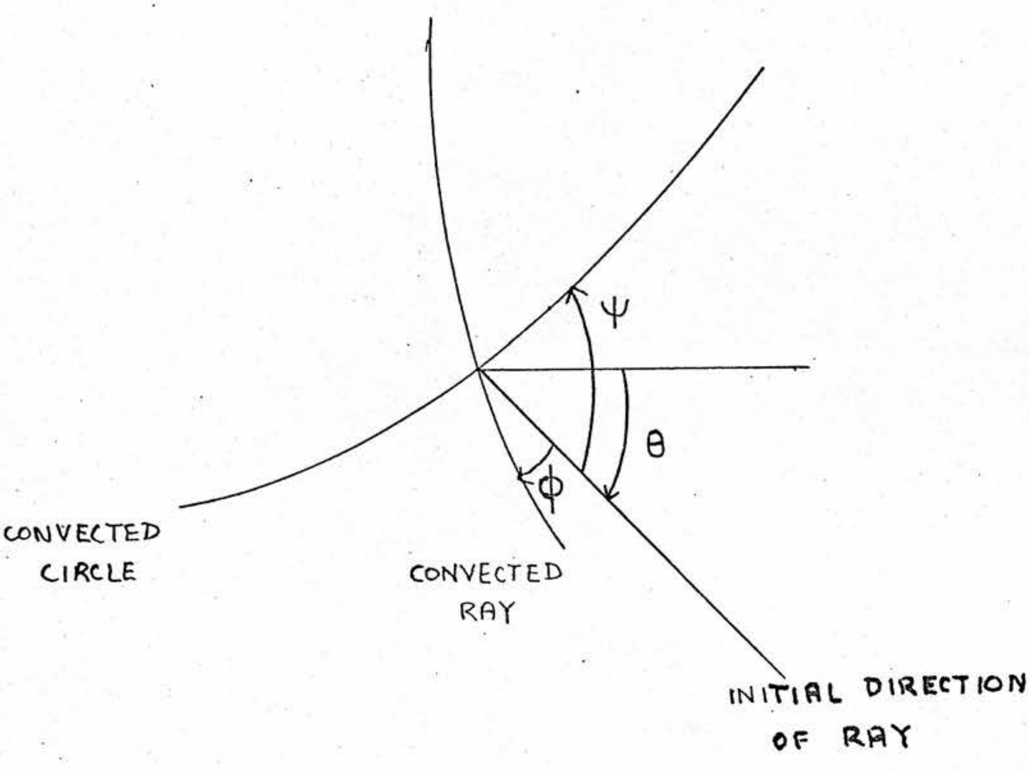


FIG. 3.10.2

Definition of the angles ϕ and ψ

The relations (3.10.3) and (3.10.4) tell us what happens to the convected rays and circles. The rays become curves which make an angle ϕ with their initial direction and $\phi = B/2$, while the angle between the convected rays and circles at any point is $(\psi - \phi)$ and this is given immediately since $A_\theta/\rho^2 = \cot(\psi-\phi)$. It is also possible to write A_ρ in terms of ϕ and ψ.

From (3.9.18) $\quad B_\rho = \dfrac{A_\rho}{\rho}\dfrac{\partial}{\partial\rho}\left(\dfrac{A_\theta}{\rho A_\rho}\right)$

$$\Rightarrow \quad B_\rho = \frac{A_{\rho\theta}}{\rho^2} - \frac{A_\theta}{\rho^3} - \frac{A_\theta A_{\rho\rho}}{\rho^2 A_\rho} .$$

From (3.10.4) $\quad A_\theta = \rho^2 \cot(\psi - \phi)$

$$\Rightarrow \quad A_{\rho\theta} = 2\rho \cot(\psi - \phi) - \rho^2 \operatorname{cosec}^2(\psi - \phi)(\psi_\rho - \phi_\rho) .$$

Hence, since $B = 2\phi$,

$$2\phi_\rho = \frac{1}{\rho}\cot(\psi-\phi) - \operatorname{cosec}^2(\psi-\phi)(\psi_\rho-\phi_\rho) - \cot(\psi-\phi)\frac{A_{\rho\rho}}{A_\rho}$$

$$\Rightarrow A_\rho = \rho \cot(\psi-\phi) \exp\left\{ -2\int^\rho \phi_\rho \tan(\psi-\phi)d\rho \right\}. \qquad (3.10.5)$$

The integral contains an arbitrary function of θ which has to be chosen so that $A_\rho/\rho \to 1$ as $\rho \to \infty$. The asymptotic behaviour of ϕ and ψ can easily be deduced since the asymptotic behaviour of A and B is known. From (3.9.31) and (3.9.32) we know that

$$A \sim \frac{\rho^2}{2} - \frac{4m \sin\theta}{3\rho} \quad \text{and} \quad B \sim -\frac{4m\cos\theta}{3\rho^3} .$$

It follows at once from (3.10.3) that

$$\phi \sim -\frac{2m\cos\theta}{3\rho^3} . \qquad (3.10.6)$$

Since $A_\theta \sim -\dfrac{4m\cos\theta}{3\rho}$ it follows from (3.10.4)

that/

that

$$\psi - \phi \sim \frac{\pi}{2} + \frac{4m \cos \theta}{3\rho^3}$$

$$\Rightarrow \quad \psi \sim \frac{\pi}{2} + \frac{2m \cos \theta}{3\rho^3} \qquad (3.10.7)$$

$$\Rightarrow \quad \psi \sim \frac{\pi}{2} - \phi .$$

Clearly as $\rho \to \infty$, $\phi \to 0$ and $\psi \to \pi/2$ as expected. In addition we can say that for large ρ, ϕ and $(\psi - \pi/2)$ are both changed by the same small amount as shown in Figure 3.10.3. The angle between the curves is not preserved but is augmented by an amount equal to double the change in ϕ. That is to say, $\psi - \phi \sim \frac{\pi}{2} - 2\phi$

$$\Rightarrow \quad \psi - \phi \sim \frac{\pi}{2} + \frac{4m \cos \theta}{3\rho^3} . \qquad (3.10.8)$$

We are now in a position to realise the significance of some of the boundary conditions obtained earlier for A and B. From (3.9.29) $A_\theta = 0$ and $B = 0$ on $\theta = -\pi/2, \rho > \ell$.

Since $B = 2\phi$, $\phi = 0$ on $\theta = -\pi/2$, $\rho > \ell$. (3.10.9)

This is only to be expected, for it says that the line $\theta = -\pi/2, \rho > \ell$ suffers no change of direction.

Since $A_\theta = \rho^2 \cot (\psi - \phi)$, $\psi = \pi/2$ on $\theta = -\pi/2, \rho > \ell$. (3.10.10)

This tells us that, for $\rho > \ell$, the convected circles continue to cut the axis of symmetry at right angles. This in no way contradicts the asymptotic behaviour predicted by (3.10.8) because in this case $\cos\theta = 0$ and the angle $\pi/2$ between the rays and circles is indeed preserved for large ρ.

Now, on $\theta = -\pi/2$, $0 < \rho < \ell$, we have from (3.9.28)

A/

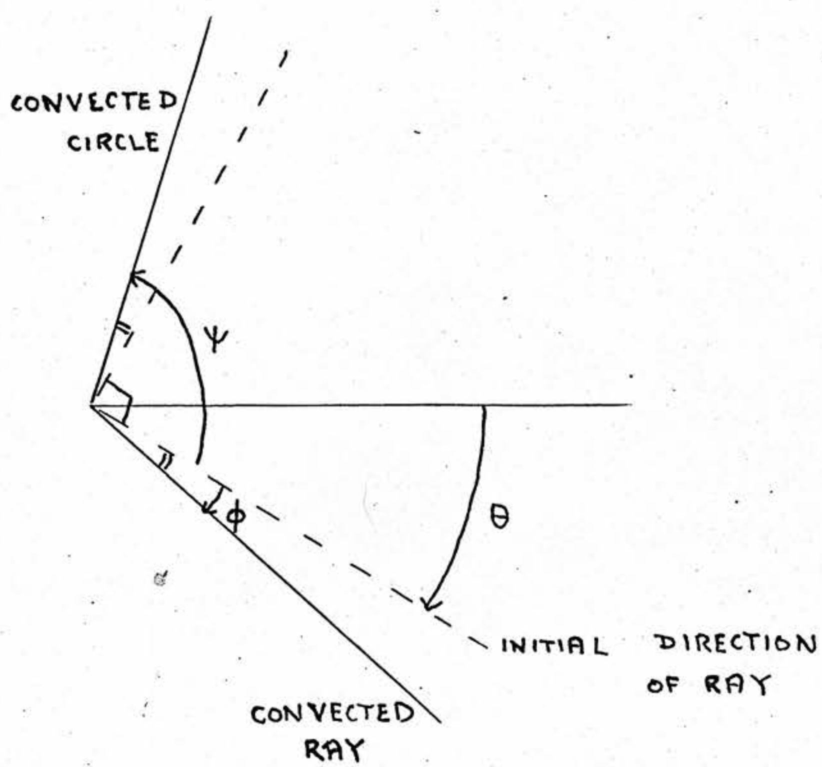


FIG. 3.10.3

The angles ϕ and ψ for large values of ρ

$$A_\theta + K\rho A_\rho = 0 \text{ and } B = -2\alpha \quad .$$

Hence $\phi = -\alpha$ on $\theta = -\pi/2$, $0 < \rho < \ell$. (3.10.11)

Again this is what is to be expected since the line segment for which $\theta = -\pi/2$, $0 < \rho < \ell$, clearly makes an angle $-\alpha$ with its initial direction.

Since $\phi = -\alpha$, $A_\theta = \rho^2 \cot(\alpha+\psi)$ and $A_\rho = -(\rho/K)\cot(\alpha+\psi)$. (3.10.12)

It is a simple matter to reproduce the earlier results relating the slope of the normal line elements to the slope of the free surface.

From (3.9.27) we know that $A_\rho = \rho$ on $\theta = 0$.

Since $B_\rho = (A_\rho/\rho)\, \partial/\partial\rho\, (A_\theta/\rho A_\rho)$, $B = A_\theta/\rho^2$ on $\theta = 0$.

But $B = 2\phi$ and $A_\theta/\rho^2 = \cot(\psi-\phi)$.

Hence $2\phi = \cot(\psi-\phi)$ on $\theta = 0$.

$$\Rightarrow \tan\psi = \frac{1 + 2\phi\tan\phi}{2\phi - \tan\phi} \quad . \tag{3.10.13}$$

This is identical with (3.8.1) as expected since the new definitions of ϕ and ψ are merely a generalisation of the original ones introduced in section (3.8) to which they reduce on the free surface.

CHAPTER 4.

(4.1) Introduction.

This chapter is concerned with a local analysis of the behaviour of the fluid in two particular problems. Firstly, an analysis is made of the behaviour of the fluid in the neighbourhood of the contact point in the case of the two-dimensional wedge water-entry problem. Secondly, the three-dimensional aperture problem is considered and in particular the behaviour of the fluid near the tip of the jet is studied. Eulerian variables are employed throughout this chapter.

(4.2) Flow near the contact point of a free surface and a solid surface.

We shall consider the two-dimensional symmetric water-entry problem only, although the method may be applied more widely. The notation used will be similar to that employed in Chapter 2 (see also Figure 4.2.1). It is assumed that the contact angle is non-zero. This is consistent with the results proved by Mackie (1969) and Garabedian (1965). Let the length of AB be k. Let $x = k \sin \alpha + \xi$ and let $y = k \cos \alpha - 1 + \eta$. We know that $\partial\phi/\partial x$ and $\partial\phi/\partial y$ are finite and non-zero at B and moreover that $\nabla\phi$ has a stationary value at B (see Mackie (1969)).

Let/

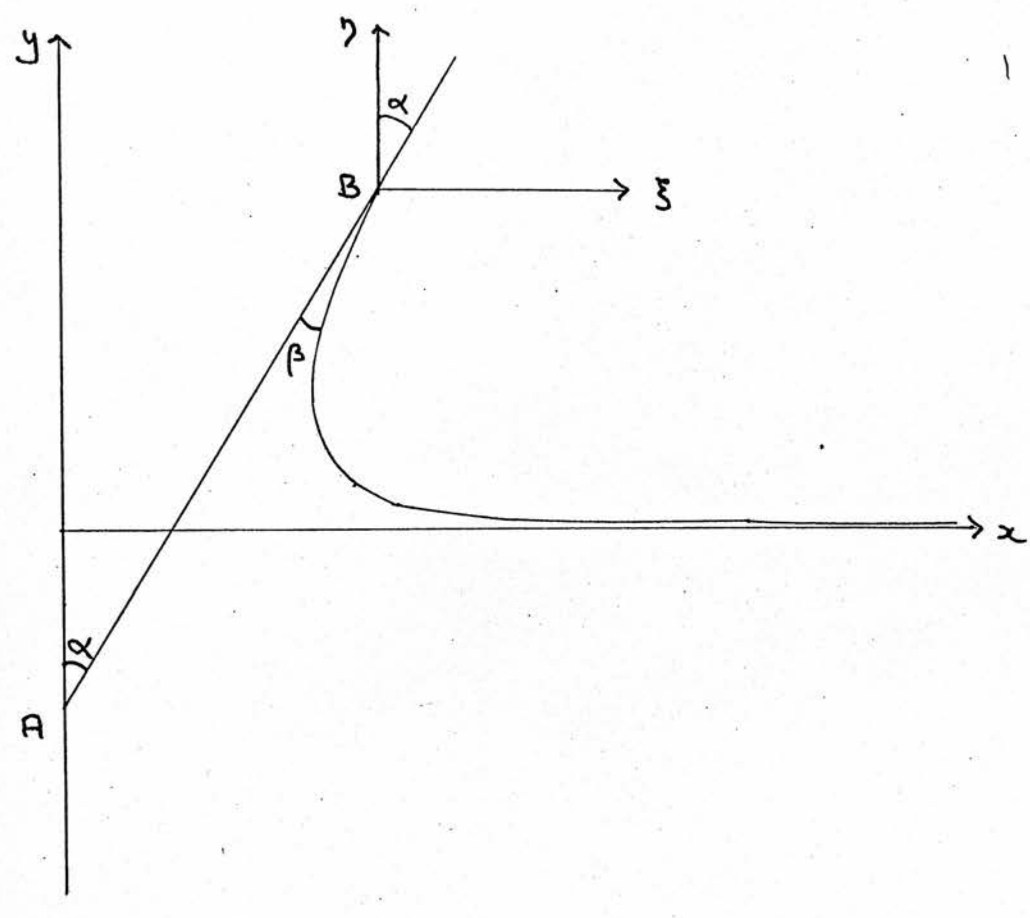


FIG. 4.2.1

Water entry of a wedge

Let $\zeta = \xi + i\,\eta$ and let $w(\zeta) = \phi + i\psi$.

Then $$\frac{dw}{d\zeta} = \frac{\partial\phi}{\partial\xi} - i\,\frac{\partial\phi}{\partial\eta}$$

$$= k\sin\alpha - i(k\cos\alpha - 1) \quad \text{at} \quad B.$$

Since $d^2w/d\zeta^2 = 0$ at B we can write

$$w(\zeta) = c_o + \{k\sin\alpha - i(k\cos\alpha - 1)\}\zeta + c_a\,\zeta^a + o(\zeta^a)$$

near the contact point, where $a > 2$ and c_o and c_a are complex constants. Then

$$\phi = \mathrm{Re}(c_o) + (k\cos\alpha - 1)\eta + k\xi\sin\alpha + \mathrm{Re}(c_a)(\xi^2+\eta^2)^{a/2}\cos a(\tan^{-1}\frac{\eta}{\xi})$$

$$- \mathrm{Im}(c_a)(\xi^2+\eta^2)^{a/2}\sin a(\tan^{-1}\frac{\eta}{\xi}) + o(\xi^2+\eta^2)^{a/2} . \qquad (4.2.1)$$

Now on the free surface the pressure is zero and so

$$(\frac{\partial\phi}{\partial x})^2 + (\frac{\partial\phi}{\partial y})^2 - 2x\,\frac{\partial\phi}{\partial x} - 2y\,\frac{\partial\phi}{\partial y} + 2\phi = 0 .$$

At B, $\partial\phi/\partial x = k\sin\alpha$ and $\partial\phi/\partial y = k\cos\alpha - 1$, and so

$2\phi = k^2 - 2k\cos\alpha + 1$ at B. Then (4.2.1) becomes

$$\phi = \tfrac{1}{2}(k^2 - 2k\cos\alpha + 1) + (k\cos\alpha - 1)\eta + k\xi\sin\alpha + \mathrm{Re}(c_a)(\xi^2+\eta^2)^{a/2}\cos a(\tan^{-1}\frac{\eta}{\xi})$$

$$- \mathrm{Im}(c_a)(\xi^2+\eta^2)^{a/2}\sin a(\tan^{-1}\frac{\eta}{\xi}) + o(\xi^2+\eta^2)^{a/2} . \qquad (4.2.2)$$

Let $\chi = \phi - \tfrac{1}{2}(x^2 + y^2)$. Then from (4.2.2)

$$\chi = -\tfrac{1}{2}r^2 + A_1 r^a\cos a\theta + B_1 r^a\sin a\theta + o(r^a) , \qquad (4.2.3)$$

where $r^2 = \xi^2+\eta^2$, $\theta = \tan^{-1}(\eta/\xi)$, $A_1 = \mathrm{Re}(c_a)$ and $B_1 = -\mathrm{Im}(c_a)$.

There are three boundary conditions to be satisfied and these are given by (2.4.2), (2.4.4) and (2.4.5).

On AB, $$\frac{\partial\phi}{\partial x}\cos\alpha - \frac{\partial\phi}{\partial y}\sin\alpha = \sin\alpha$$

$$\Rightarrow \frac{\partial\chi}{\partial\xi}\cos\alpha = \frac{\partial\chi}{\partial\eta}\sin\alpha \quad \text{on} \quad \eta = \xi\cot\alpha . \qquad (4.2.4)$$

On the free surface, Bernoulli's equation gives

$\partial/$

$$(\frac{\partial\phi}{\partial x})^2 + (\frac{\partial\phi}{\partial y})^2 - 2x\,\frac{\partial\phi}{\partial x} - 2y\,\frac{\partial\phi}{\partial y} + 2\phi = 0$$

$$\Rightarrow (\frac{\partial\chi}{\partial\xi})^2 + (\frac{\partial\chi}{\partial\eta})^2 + 2\chi = 0 \quad . \qquad (4.2.5)$$

Also, if the equation of the free surface is $\xi = h(\eta)$, then $\partial\chi/\partial\xi = h'(\eta)\,\partial\chi/\partial\eta$ and it follows that, on $\xi = h(\eta)$, $(\nabla\chi)^2 = (d\chi/ds)^2$ where s is arc-length measured along the free surface. Then (4.2.5) becomes

$$(\frac{d\chi}{ds})^2 + 2\chi = 0 \ .$$

Integration gives $\chi = -\frac{1}{2}s^2$ if $s = 0$ at B and $\chi = 0$ there from (4.2.3). On the free surface there are thus two conditions to be satisfied and these may be taken to be

$$\chi = -\tfrac{1}{2}s^2 \qquad (4.2.6)$$

$$\frac{\partial\chi}{\partial\xi} = h'(\eta)\,\frac{\partial\chi}{\partial\eta} \quad . \qquad (4.2.7)$$

Differentiation of (4.2.3) gives

$$\frac{\partial\chi}{\partial\xi} = -r\cos\theta + A_1 a\, r^{a-1}\cos(a-1)\theta + B_1 a\, r^{a-1}\sin(a-1)\theta + o(r^{a-1})$$

$$\frac{\partial\chi}{\partial\eta} = -r\sin\theta - A_1 a\, r^{a-1}\sin(a-1)\theta + B_1 a\, r^{a-1}\cos(a-1)\theta + o(r^{a-1}) \ .$$

In order to satisfy (4.2.4) we must have

$$\frac{\partial\chi}{\partial\xi}\cos\alpha = \frac{\partial\chi}{\partial\eta}\sin \qquad \text{on} \quad \theta = -(\pi/2 + \alpha) \ .$$

Hence $B_1 = -A_1\tan a(\pi/2+\alpha)$. Let $A_1 = D_1\cos a(\pi/2+\alpha)$ and let $B_1 = -D_1\sin a(\pi/2+\alpha)$. Then

$$\chi = -\tfrac{1}{2}r^2 + D_1 r^a \cos a\,(\frac{\pi}{2} + \alpha + \theta) + o(r^a) \ . \qquad (4.2.8)$$

On the free surface (4.2.6) and (4.2.7) must be satisfied.

Let $h(\eta) = -\eta\tan(\beta-\alpha) + c_m\,\eta^m + o(\eta^m)$ (4.2.9)

where $m > 1$ and $c_m \neq 0$, since it is assumed that $\beta \neq 0$.

We may calculate s^2 using (4.2.9) since

$$s = \int_0^\eta \{1+(h'(\eta))^2\}^{\frac{1}{2}}\,d\eta \ .$$

Hence/

Hence $\quad s = \sec(\beta-\alpha)\,\{\eta - \dfrac{c_m\tan(\beta-\alpha)}{\sec^2(\beta-\alpha)}\,\eta^m + o(\eta^m)\,\}$

$$\Rightarrow s^2 = \eta^2\sec^2(\beta-\alpha) - 2c_m\tan(\beta-\alpha)\eta^{m+1} + o(\eta^{m+1})\,. \tag{4.2.10}$$

On $\xi = h(\eta)$, $\xi^2 + \eta^2 = \eta^2\sec^2(\beta-\alpha) - 2c_m\tan(\beta-\alpha)^{m+1} + o(\eta^{m+1})$

$$\Rightarrow (\xi^2+\eta^2)^{a/2} = \eta^a\sec^a(\beta-\alpha) + o(\eta^a)\,.$$

Hence on $\xi = h(\eta)$, from (4.2.8),

$$\chi = -\tfrac{1}{2}(\eta^2\sec^2(\beta-\alpha) - 2c_m\tan(\beta-\alpha)\eta^{m+1} + o(\eta^{m+1})\,)$$
$$+ D_1\,\eta^a\sec^a(\beta-\alpha)\cos a\beta + o(\eta^a)\,. \tag{4.2.11}$$

It is necessary to include both the terms $o(\eta^a)$ and $o(\eta^{m+1})$ since, although we know that $a > 2$ and $m > 1$, we do not know at this stage whether a is greater than, equal to or less than $m+1$. In order to satisfy (4.2.6) we have from (4.2.10) and (4.2.11)

$$-\tfrac{1}{2}(\eta^2\sec^2(\beta-\alpha) - 2c_m\tan(\beta-\alpha)\eta^{m+1}) + D_1\,\eta^a\sec^a(\beta-\alpha)\cos a\beta$$
$$= -\tfrac{1}{2}(\eta^2\sec^2(\beta-\alpha) - 2c_m\tan(\beta-\alpha)\eta^{m+1}) + o(\eta^{m+1}) + o(\eta^a)\,.$$

Hence, either $a > m+1$ or $a \leqslant m+1$ and $\cos a\beta = 0$. (4.2.12)

Now from (4.2.8),

$$\frac{\partial\chi}{\partial\xi} = -\xi + D_1 a\,r^{a-1}\cos(a(\tfrac{\pi}{2}+\alpha)+(a-1)\theta) + o(r^{a-1})$$

$$\frac{\partial\chi}{\partial\eta} = -\eta - D_1 a\,r^{a-1}\sin(a(\tfrac{\pi}{2}+\alpha)+(a-1)\theta) + o(r^{a-1})\,.$$

Hence, on $\xi = h(\eta)$,

$$\frac{\partial\chi}{\partial\xi} = \eta\tan(\beta-\alpha) - c_m\eta^m + D_1 a\eta^{a-1}\sec^{a-2}(\beta-\alpha)\left\{-\tan(\beta-\alpha)\cos a\beta + \sin a\beta\right\}$$
$$+ o(\eta^m) + o(\eta^{a-1})\,,$$

$$\frac{\partial\chi}{\partial\eta} = -\eta + D_1 a\eta^{a-1}\sec^{a-2}(\beta-\alpha)\left\{\cos a\beta + \tan(\beta-\alpha)\sin a\beta\right\} + o(\eta^{a-1})\,.$$

In/

In order to satisfy (4.2.7),

$$\eta\tan(\beta-\alpha)-mc_m\eta^m+D_1a\eta^{a-1}\sec^{a-2}(\beta-\alpha)\{-\tan(\beta-\alpha)\cos a\beta-\tan^2(\beta-\alpha)\sin a\beta\}$$
$$=\eta\tan(\beta-\alpha)-c_m\eta^m+D_1a\eta^{a-1}\sec^{a-2}(\beta-\alpha)\{-\tan(\beta-\alpha)\cos a\beta+\sin a\beta\}$$
$$+\,o(\eta^m)+o(\eta^{a-1})\ .$$

Since $m \neq 1$ and $c_m \neq 0$, we must have $m = a-1$ and from (4.2.12), $\cos a\beta = 0$. (4.2.13)

Then $$c_m = -\frac{D_1a\sec^a(\beta-\alpha)\sin a\beta}{a-2}\ . \qquad (4.2.14)$$

Since $\cos a\beta = 0$, $a\beta = \pi/2 + n\pi$ for some integer n. We shall dispose of every possibility except $n = 0$ on the hypothesis that the pressure is positive in the interior of the fluid. The dimensionless pressure p is given by Bernoulli's equation in the form

$$\tfrac{1}{2}(\nabla\chi)^2 + \chi + p = 0\ . \qquad (4.2.15)$$

From (4.2.8) we see that $(\nabla\chi)^2 = r^2 - 2D_1a\, r^a\cos a(\frac{\pi}{2}+\alpha+\theta)$. (4.2.15) then yields

$$p = (a-1)\,D_1r^a\cos a(\frac{\pi}{2}+\alpha+\theta)\ . \qquad (4.2.16)$$

Now in the region of fluid under consideration, $0 \leqslant \pi/2 + \alpha + \theta \leqslant \beta$, and on $\pi/2 + \alpha + \theta = 0$, $p = (a-1)D_1r^a$ whereas on $\pi/2 + \alpha + \theta = \beta$, $p = 0$. If the pressure is positive within the fluid then $\cos a(\pi/2 + \alpha + \theta) > 0$ for $0 < \pi/2+\alpha+\theta < \beta$ and so $a\,\alpha\beta \leqslant \pi/2$. But we know from (4.2.13) that $\cos a\beta = 0$ and so the only possibility is $a\beta = \pi/2$. This gives the familiar result $\beta < \pi/4$, since $a > 2$. The assumption that the pressure is positive in the interior of the fluid has been shown by Mackie (1969) to be a reasonable one (see also section 3.8). Another interesting point is that Garabedian (1953) showed that for

a/

a certain unsymmetric water-entry problem a contact angle of $\pi/2$ arose, and this is now seen to be incompatible with the hypothesis of positive pressure. Although the present analysis was approached from the point of view of the symmetric problem, symmetry properties were not in fact used and the results only depend on three basic hypotheses - similarity, conservation of arc-length on the free surface, and the fact that the same particle remains at the contact point throughout the motion. These are all satisfied in Garabedian's problem. The conclusion is that in that problem the pressure is negative on the wedge face, and indeed over a certain finite region of fluid. The simplest solution of $\cos(a\pi/2) = 0$ with $a > 2$ is $a = 3$ and then the pressure is negative near the contact point for $0 \leqslant \pi/2 + \alpha + \theta \leqslant \pi/6$. If any other solution is chosen then further regions of negative pressure occur.

To return to the symmetric water-entry problem, we have shown that, near the contact point,

$$\chi = -\tfrac{1}{2}r^2 + D_1 r^a \cos a(\pi/2 + \alpha + \theta) + o(r^a)$$

and $\quad h(\eta) = -\eta \tan(\beta-\alpha) + c_m \eta^{a-1} + o(\eta^{a-1})$

where $a = \pi/2\beta$ and $c_m = -D_1 a \sec^a(\beta-\alpha)/(a-2)$.

Moreover $\beta < \pi/4$ provided the pressure is positive in the interior and since $h''(\eta) = c_m(a-1)(a-2)\eta^{a-3} + o(\eta^{a-3})$ we see that the curvature is zero, finite or infinite at the contact point according as the contact angle is less than, equal to, or greater than $\pi/6$. This confirms Mackie's result (see also section 2.4).

The most interesting feature about the above analysis is that the result $\beta < \pi/4$ (and the associated curvature result) has been shown to be an essentially local property depending only/

only on self-similarity and free surface properties. Earlier proofs of these results used properties of the global solution of the symmetric problem and were therefore only applicable to such symmetric cases. Moreover these proofs depended either directly or indirectly on the assumption that the pressure is non-negative within the fluid, and this is the only global property used in the present case.

It is possible to calculate a second term in the expansions for χ and $h(\eta)$. The details are somewhat tedious and are given in Appendix III.
The results are set out below.

$$\chi = -\tfrac{1}{2}r^2 + D_1 r^a \cos a(\pi/2+\alpha+\theta) + D_2 r^{2a-2} \cos 2(a-1)(\pi/2+\alpha+\theta) + o(r^{2a-2}),$$

and $h(\eta) = -\eta\tan(\beta-\alpha) + c_m\eta^{a-1} + c_n\eta^{2a-3} + o(\eta^{2a-3})$,

where $D_2 \sec^{2a}(\beta-\alpha)\cos 2\beta = -\tfrac{1}{2} c_m{}^2 \dfrac{(3a-4)(a-2)}{(2a-3)}$

and $c_n = \dfrac{\cos^2(\beta-\alpha)c_m{}^2}{2}\left\{\dfrac{(a-1)(3a-4)}{(2a-3)}\tan 2\beta - 2(a-1)\tan(\beta-\alpha)\right\}$.

After the above work was completed, the author learned of a recent study by Dr. A.B. Tayler of flow near the contact point of a free and a solid surface. In this work, which is not yet published, Tayler considers the general problem of flow in the neighbourhood of the separation point of a free surface and a solid boundary without any simplifying assumptions of similarity or symmetry. He enumerates the various cases which may occur - these include Riabouchinsky flows, the problem of waves on a sloping beach and the water-entry problem, as well as other possibilities for which no global solution is known. His results for/

for the wedge water-entry problem confirm that the contact angle is less than $\pi/4$ and also Mackie's result concerning the curvature of the free surface at the separation point.

(4.3) The aperture problem.

It is difficult to apply the arguments of the previous section to the three-dimensional cone water-entry problem since it is not obvious what type of harmonic functions should be used near a point which is not on the axis of symmetry. However the aperture problem is a three-dimensional problem and the tip of the jet lies on the axis of symmetry so that we can look for a solution near this point in terms of Legendre functions. In the aperture problem, water is initially confined at constant pressure P_o in the half-space $Z \geqslant 0$. At time $t = 0$ a circular aperture with centre the origin forms in the boundary plane $Z = 0$ and its radius increases at the constant rate V_o. Then an axially symmetric jet of water comes out of the hole thus produced. If R and Z are cylindrical polar coordinates, then similarity variables ρ and z may be defined by $\rho = R/V_o t$ and $z = Z/V_o t$. Let the equation of the free surface be $z = f(\rho)$. Since the pressure is zero on the free surface, then in the meridian plane, arc-length is conserved on the free surface and the free surface is convex towards the fluid, provided the pressure is positive in the interior of the fluid. (The reasoning which establishes these results is identical to that used in the two-dimensional wedge water-entry problem). The dimensionless velocity potential satisfies Laplace's equation

∂/

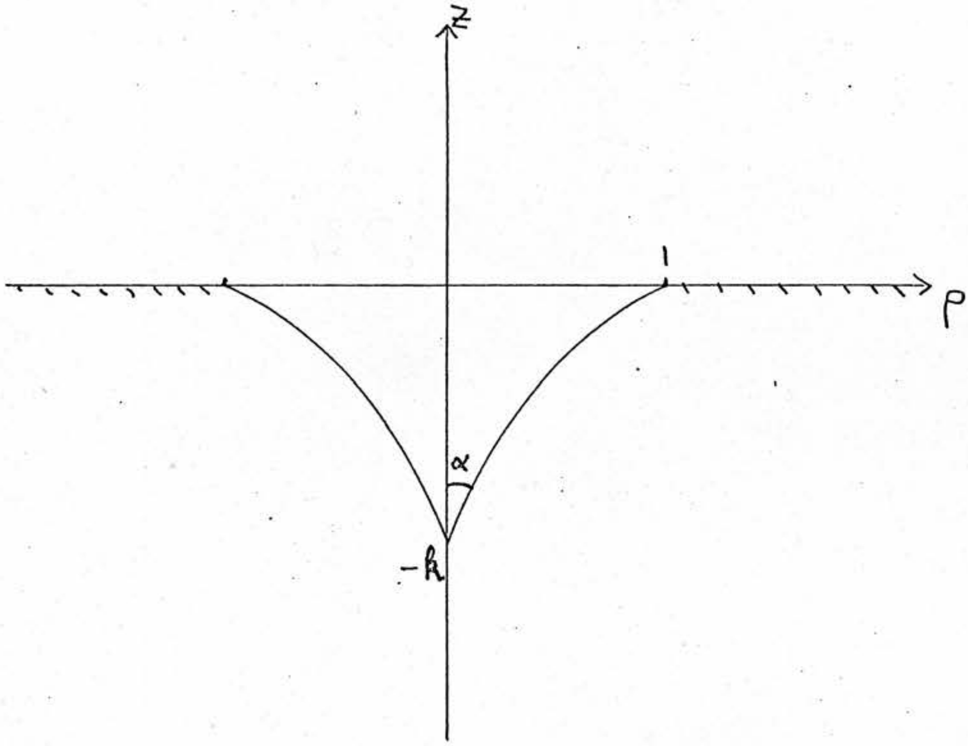


FIG. 4.3.1

The aperture problem

$$\frac{\partial^2\phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\phi}{\partial\rho} + \frac{\partial^2\phi}{\partial z^2} = 0 . \qquad (4.3.1)$$

The boundary conditions are:

On $z = 0, \ \rho > 1, \quad \frac{\partial\phi}{\partial z} = 0$. (4.3.2)

On $z = f(\rho), \ \frac{\partial\phi}{\partial z} - z = f'(\rho)\left(\frac{\partial\phi}{\partial\rho} - \rho\right)$. (4.3.3)

On $z = f(\rho), \ \frac{1}{2}\left(\frac{\partial\phi}{\partial\rho}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial z}\right)^2 - \rho\frac{\partial\phi}{\partial\rho} - z\frac{\partial\phi}{\partial z} + \phi = p_o$ (4.3.4)

where $p_o = P_o/\rho_o V_o$ and ρ_o is the density of the fluid, and provided $\phi \to 0$ as $\rho \to \infty$.

(4.4) Behaviour near the tip of the jet.

Let $x = \rho$ and let $y = z+k$. Let $\chi = \phi - \frac{1}{2}(\rho^2 + z^2)$.
Now on the free surface $\frac{1}{2}(\nabla\phi)^2 - \rho\phi_\rho - z\phi_z + \phi = p_o$.
At$(0,-k)$, $\phi_\rho = 0$ and $\phi_z = -k$, and so $\phi(0,-k) = p_o + \frac{1}{2}k^2$.
Let $\phi = p_o + \frac{1}{2}k^2 - ky + cr^a P_a(\cos\theta) + o(r^a)$ near $(0,-k)$ where $y = r\cos\theta$ and $x = r\sin\theta$, c is a constant to be found, and

$$P_a(s) = \frac{1}{2\pi i}\int_\gamma \frac{(t^2-1)^a}{2^a\,(t-s)^{a+1}}\,dt$$

where the t-plane is cut along the real axis from -1 to $-\infty$ and along an arbitrary path from the point 1 to the point s and the s-plane is cut analogously. The contour γ is a closed curve, oriented in the positive sense, which encloses the points $t = s$ and $t = 1$ but excludes the point $t = -1$.

Then $\chi = p_o - \frac{1}{2}(x^2 + y^2) + cr^a P_a(\cos\theta) + o(r^a) + o(r^2)$. (4.4.1)

The inclusion of both the terms $o(r^a)$ and $o(r^2)$ is necessary since we do not know whether $a < 2$ or $a \geqslant 2$.

Let/

Let the equation of the free surface be given by

$$y = g(x) = x \cot \alpha + c_m x^m + o(x^m) . \qquad (4.4.2)$$

There are two conditions to be satisfied on $y = g(x)$:

$$\frac{\partial \chi}{\partial y} = g'(x) \frac{\partial \chi}{\partial x} \qquad (4.4.3)$$

$$\chi = p_o - \tfrac{1}{2} s^2 \qquad (4.4.4)$$

where s is arc-length measured along the free surface.

The second condition follows from (4.3.4) and the fact that

$\nabla\chi = d\chi/ds$ on the free surface. Suppose first that $a < 2$.

Then (4.4.1) may be written as

$$\chi = p_o + c\, r^a P_a(\cos\theta) + o(r^a) . \qquad (4.4.5)$$

Then $\frac{\partial \chi}{\partial x} = c\, r^{a-1} \{a \sin\theta\, P_a(\cos\theta) - \sin\theta\cos\theta\, P_a'(\cos\theta)\} + o(r^{a-1})$

and $\frac{\partial \chi}{\partial y} = c\, r^{a-1} \{a \cos\theta\, P_a(\cos\theta) + \sin^2 P_a'(\cos\theta)\} + o(r^{a-1})$.

On $y = g(x)$, $r^{a-1} = x^{a-1} \operatorname{cosec}^{a-1}\alpha + o(x^{a-1})$.

Hence, on $y = g(x)$,

$$\frac{\partial \chi}{\partial x} = cx^{a-1}\operatorname{cosec}^{a-1}\alpha\{a \sin\alpha\, P_a(\cos\alpha) - \sin\alpha\cos\alpha P_a'(\cos\alpha)\} + o(x^{a-1})$$

$$\frac{\partial \chi}{\partial y} = cx^{a-1}\operatorname{cosec}^{a-1}\alpha\{a \cos\alpha\, P_a(\cos\alpha) + \sin^2\alpha\, P_a'(\cos\alpha)\} + o(x^{a-1}) .$$

In order to satisfy (4.4.3) we must have

$$cx^{a-1} \operatorname{cosec}^{a-1}\alpha \{a \cos\alpha\, P_a(\cos\alpha) + \sin^2\alpha\, P_a'(\cos\alpha)\}$$
$$= cx^{a-1}\operatorname{cosec}^{a-1}\alpha\{a\cos\alpha P_a(\cos\alpha) - \cos^2\alpha\, P_a'(\cos\alpha)\} + o(x^{a-1}) + O(x^{a+m-2}) .$$

$$\Rightarrow P_a'(\cos\alpha) = 0 . \qquad (4.4.6)$$

The next step is to calculate s^2 on $y = g(x)$.

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \frac{ds}{dx} = \operatorname{cosec}\alpha\left(1 + \frac{m\, c_m \cot\alpha}{\operatorname{cosec}^2\alpha}\, x^{m-1} + o(x^{m-1})\right)$$

=>/

$$\Rightarrow s \;=\; x \operatorname{cosec} \alpha + \frac{c_m \cot \alpha}{\operatorname{cosec} \alpha}\; x^m + o(x^m)$$

$$\Rightarrow s^2 \;=\; x^2 \operatorname{cosec}^2 \alpha + 2c_m \cot \alpha\; x^{m+1} + o(x^{m+1}) \;. \qquad (4.4.7)$$

In order to satisfy (4.4.4) we must have

$$p_o + cx^a \operatorname{cosec}^a \alpha\; P_a(\cos\alpha) = p_o - \tfrac{1}{2}(x^2 \operatorname{cosec}^2 \alpha + 2c\, x^{m+1} \cot \alpha) + o(x^{m+1}) + o(x^a) \;.$$

But we have assumed that $a < 2$, and we know that $m+1 > 2$ since $m > 1$ and so

$$P_a(\cos \alpha) \;=\; 0 \;. \qquad (4.4.8)$$

Now (4.4.6) and (4.4.8) cannot both be satisfied unless $P_a(\cos \theta) \equiv 0$, which is impossible. It must be concluded therefore that $a \geqslant 2$. Since $a \geqslant 2$, (4.4.1) may be written as

$$\chi \;=\; p_o - \tfrac{1}{2}(x^2 + y^2) + c\, r^a\, P_a(\cos \theta) + o(r^a) \;. \qquad (4.4.9)$$

Now $\dfrac{\partial\chi}{\partial x} = -x + cr^{a-1}\{a \sin\theta\, P_a(\cos\theta) - \sin\theta\cos\theta\, P_a'(\cos\theta)\} + o(r^{a-1})$

and $\dfrac{\partial\chi}{\partial y} = -y + cr^{a-1}\{a \cos\theta\, P_a(\cos\theta) + \sin^2\theta\, P_a'(\cos\theta)\} + o(r^{a-1})$.

On $y = g(x)$,

$$\frac{\partial\chi}{\partial x} = -x + cx^{a-1} \operatorname{cosec}^{a-1}\alpha\{a\sin\alpha\, P_a(\cos\alpha) - \sin\alpha \cos\alpha\, P_a'(\cos\alpha)\} + o(x^{a-1})$$

$$\frac{\partial\chi}{\partial y} = -x \cot\alpha - c_m x^m + cx^{a-1} \operatorname{cosec}^{a-1}\alpha\{a \cos\alpha\, P_a(\cos\alpha) + \sin^2\alpha\, P_a'(\cos\alpha)\} + o(x^m) + o(x^{a-1}) \;.$$

In order to satisfy (4.4.3) we must have

$$-x \cot \alpha - c_m x^m + cx^{a-1} \operatorname{cosec}^{a-1}\alpha\{a \cos\alpha\, P_a(\cos \alpha) + \sin^2\alpha\, P_a'(\cos\ )\}$$

$$= -x\cot\alpha - mc_m x^m + cx^{a-1} \operatorname{cosec}^{a-1}\alpha\{a \cos\alpha\, P_a(\cos \alpha) - \cos^2\alpha\, P_a'(\cos\alpha)\} + o(x^{a-1}) + o(x^m) + O(x^{a+m-2}) \;.$$

Since $m \neq 1$, $c_m \neq 0$, either

$m > a-1$ and $P_a'(\cos \alpha) = 0$ or

$$m = a-1 \text{ and } c_m = -\, c \operatorname{cosec}^{a-1}\alpha\, P_a'(\cos \alpha)/(a-2) \;. \qquad (4.4.10)$$

Next/

Next we must satisfy (4.4.4). On $y = g(x)$,

$x^2 + y^2 = x^2\operatorname{cosec}^2\alpha + 2c_m \cot\alpha\, x^{m+1} + o(x^{m+1})$ and s^2 is still given by (4.4.7). Hence (4.4.4) becomes

$$p_o - \tfrac{1}{2}(x^2\operatorname{cosec}^2\alpha + 2c_m\cot\alpha\, x^{m+1}) + c\, x^a\operatorname{cosec}^a\alpha\, P_a(\cos\alpha)$$

$$= p_o - \tfrac{1}{2}(x^2\operatorname{cosec}^2\alpha + 2\, c_m\cot\alpha\, x^{m+1}) + o(x^{m+1}) + o(x^a)\ .$$

From (4.4.10) we know that $a \leqslant m+1$ and to satisfy (4.4.4) the only possibility is

$$a = m+1 \quad\text{and}\quad P_a(\cos\alpha) = 0\ . \qquad (4.4.11)$$

Summing up, we have shown that near the tip of the jet,

$$\begin{aligned} \chi &= p_o - \tfrac{1}{2} r^2 + c\, r^a P_a(\cos\theta) + o(r^a) \\ g(x) &= x\cot\alpha + c_m\, x^{a-1} + o(x^{a-1}) \end{aligned} \qquad (4.4.12)$$

where $P_a(\cos\alpha) = 0$ and $c_m = -\, c\operatorname{cosec}^{a-1}\alpha\, P_a'(\cos\alpha)/(a-2)$.

It has been assumed in this analysis that $\alpha \neq 0$, since the behaviour in that case would be more complicated and beyond the scope of this approach. We can however give α an upper bound. Since $a = m+1$ and $m > 1$, $a > 2$. The only root of $P_2(\cos\alpha) = 0$ for $0 < \alpha < \pi/2$ is $\cos\alpha = 1/\surd 3$ and since $a > 2$ we may conclude that

$$\alpha < \cos^{-1}(1/\surd 3) \approx 54^\circ\, 44'\ .$$

This of course assumes that the dependence of α on a is continuous. By a simple local analysis bounds have been put on the angle at the tip of the jet, namely $0 \leqslant \alpha < \cos^{-1}(1/\surd 3)$, although an analysis of the entire flow field is required in order to say anything more about the values of a and α for a particular initial pressure and rate of increase of the size of the aperture.

As/

As a special case it is possible to give a linearised solution when $P_o/\rho_o V_o^2$ is small, where P_o is the initial pressure in the fluid at rest, ρ_o the density of the fluid and V_o the rate of increase of the radius of the aperture.

Let $P_o/\rho_o = U_o^2$ where U_o has the dimension of velocity.

Let $U_o/V_o = \varepsilon$. Then ε is small when the initial pressure P_o is small, or when the aperture grows quickly. Let us look for a solution in which $\phi = \varepsilon^2\bar{\phi}$ and $f(\rho) = \varepsilon^2\bar{f}(\rho)$ and terms which are $o(\varepsilon^2)$ are ignored. Then equations (4.3.1)-(4.3.4) become respectively

$$\frac{\partial^2\bar{\phi}}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\bar{\phi}}{\partial\rho} + \frac{\partial^2\bar{\phi}}{\partial z^2} = 0 , \qquad (4.4.13)$$

$$\frac{\partial\bar{\phi}}{\partial z} = 0 \text{ on } z = 0, \ \rho > 1 , \qquad (4.4.14)$$

$$\rho\bar{f}'(\rho) - \bar{f}(\rho) = -\frac{\partial\bar{\phi}}{\partial z}(\rho,0) \text{ on } z = 0, \ \rho < 1, \qquad (4.4.15)$$

$$\bar{\phi} = 1 \text{ on } z = 0 , \ \rho < 1 , \qquad (4.4.16)$$

using the fact that $\partial\phi/\partial\rho = 0$ when $\rho = 0$.

The problem posed by (4.4.13),(4.4.14) and (4.4.16) is a classical mixed boundary value problem, that of the electrified disc, and the solution may be found, for example, in Sneddon (1966), Chapter III.

$$\text{Then } \phi(\rho,z) = \frac{2\varepsilon^2}{\pi}\int_0^\infty \frac{\sin\xi}{\xi} e^{-\xi z} J_o(\rho\xi)d\xi . \qquad (4.4.17)$$

The equation of the free surface may be found from (4.4.15).

$$\text{From (4.4.17) } \frac{\partial\bar{\phi}}{\partial z} = -\frac{2}{\pi}\int_0^\infty \sin\xi \, e^{-\xi z} J_o(\rho\xi)d\xi$$

$$\Rightarrow \frac{\partial\bar{\phi}}{\partial z} = -\frac{2}{\pi}(1-\rho^2)^{-\frac{1}{2}} \text{ on } z = 0, \ \rho < 1 .$$

Hence/

Hence $\rho\bar{f}'(\rho) - \bar{f}(\rho) = \frac{2}{\pi}(1 - \rho^2)^{-\frac{1}{2}}$

$$\Rightarrow \bar{f}(\rho) = -\frac{2}{\pi}(1 - \rho^2)^{\frac{1}{2}} + \lambda\rho \quad \text{for some constant } \lambda .$$

The equation of the free surface is then

$$z = -\frac{2\varepsilon^2}{\pi}(1 - \rho^2)^{\frac{1}{2}} + \lambda\varepsilon^2\rho . \tag{4.4.18}$$

Since $f(1) = 0$ it might be concluded that $\lambda = 0$, but the linearised solution for ϕ is not valid near $\rho = 1$ and so care must be taken with this assertion.

Now, near $\rho = 0$,

$$f(\rho) = -\frac{2\varepsilon^2}{\pi}(1 - \tfrac{1}{2}\rho^2) + \lambda\varepsilon^2\rho + o(\rho^2) ,$$

and comparing this with (4.4.12) we see that $a = 3$ and $c_m = \varepsilon^2/\pi$. We must then satisfy $P_3(\cos\alpha) = 0$ and so $\alpha = \pi/2$ or $\cos^{-1}(3/5)^{\frac{1}{2}}$.

Since $f'(o) = \cot\alpha$, $\lambda\varepsilon^2 = \cot\alpha$

$$\Rightarrow \alpha = \pi/2 - \lambda\varepsilon^2 .$$

The only possibility is then $\alpha = \pi/2$ and $\lambda = 0$. The equation of the free surface is then reduced to

$$z = -\frac{2\varepsilon^2}{\pi}(1 - \rho^2)^{\frac{1}{2}} . \tag{4.4.19}$$

It can be shown that near $\rho = 0$, $z = 0$,

$$\phi \sim \varepsilon^2 - \frac{2\varepsilon^2}{\pi} r\cos\theta + \frac{2\varepsilon^2}{3\pi} r^3 P_3(\cos\theta) + o(r^3)$$

and comparison with (4.4.12) shows that $c = 2\varepsilon^2/3\pi$. It then follows that $c_m = c\,\mathrm{cosec}^2\alpha\, P_3'(\cos\alpha)$ as required.

The most obvious difficulty with this linearised solution is that the free surface, far from being convex to the fluid, actually takes up the shape of an ellipse. Moreover, it has been shown that, provided α depends continuously on a , then/

then $\alpha < \cos^{-1}(1/\sqrt{3})$, which is clearly contradicted by having $\alpha = \pi/2$. However $\alpha = \pi/2$ at least satisfies the major requirement that $P_3(\cos\alpha) = 0$. The assertion that the free surface is convex to the fluid depends on the assumption that the pressure is positive in the interior of the fluid. It is clear that the velocity at $\rho = 1$ is infinite and so the pressure is negative and infinite there. It is possible to calculate the pressure distribution on the wall and show that the pressure is in fact negative in a finite region of the fluid. The pressure is given by Bernoulli's equation, which takes the form

$$\tfrac{1}{2}\left(\frac{\partial\phi}{\partial\rho}\right)^2 + \tfrac{1}{2}\left(\frac{\partial\phi}{\partial z}\right)^2 - \rho\frac{\partial\phi}{\partial\rho} - z\frac{\partial\phi}{\partial z} + \phi + p = p_o .$$

If $\phi = \varepsilon^2\overline{\phi}$ and $p = \varepsilon^2\overline{p}$ and terms which are $o(\varepsilon^2)$ are neglected, this equation reduces to

$$\overline{p}(\rho,0) = 1 - \overline{\phi}(\rho,0) + \rho\frac{\partial\overline{\phi}}{\partial\rho}(\rho,0) \quad \text{on} \quad z = 0 . \tag{4.4.20}$$

From(4.4.17) it follows that

$$\overline{\phi}(\rho,0) = \frac{2}{\pi}\sin^{-1}\frac{1}{\rho} \quad \text{for} \quad \rho > 1$$

and $$\frac{\partial\overline{\phi}}{\partial\rho}(\rho,0) = -\frac{2}{\pi}\frac{1}{\rho}(\rho^2 - 1)^{-\frac{1}{2}} \quad \text{for} \quad \rho > 1 .$$

Hence, from (4.4.20),

$$p(\rho,0) = p_o - \frac{2\varepsilon^2}{\pi}\left\{\sin^{-1}\frac{1}{\rho} + (\rho^2 - 1)^{-\frac{1}{2}}\right\} \quad \text{for} \quad \rho > 1 .$$

Clearly $p(\rho,0) < 0$ whenever $\rho < \rho_o$ where ρ_o is given by

$$1 - (2/\pi)(\sin^{-1}(1/\rho_o) + (\rho_o^2 - 1)^{-\frac{1}{2}}) = 0 .$$

Simple calculations show that $3/2 < \rho_o < 5/3$.

For $1 < \rho < \rho_o$ the pressure on the wall is negative and it is only to be expected that the free surface is not convex to the fluid.

CHAPTER 5

(5.1) Water-entry problem for a thin wedge.

Consider the water-entry problem for a thin wedge of semi-angle ε. Let $\overline{X}(A,B,t)$ and $\overline{Y}(A,B,t)$ represent the displacements in the A and B directions of a particle which was initially at rest. Then the solution of the problem with $\varepsilon = 0$ is $\overline{X} = A$ and $\overline{Y} = B$. The free surface is the straight line $\overline{Y} = 0$ and the contact angle is $\pi/2$. It is then immediate that this is a singular perturbation problem because for any ε greater than zero, the contact angle has been shown to be less than $\pi/4$. Let us seek a solution in which $\overline{X} = A+\varepsilon X$ and $\overline{Y} = B+\varepsilon Y$ and neglect terms which are $o(\varepsilon)$.

Let $\overline{x} = \overline{X}/V_o t$, $\overline{y} = \overline{Y}/V_o t$, $x = X/V_o t$, $y = Y/V_o t$, $a = A/V_o t$ and $b = B/V_o t$. Then it follows from (3.2.9) and (3.2.10) that $\overline{x}$ and $\overline{y}$ satisfy

$$a\overline{x}_b\overline{x}_{aa} + (b\overline{x}_b - a\overline{x}_a)\overline{x}_{ab} - b\overline{x}_a\overline{x}_{bb}$$

$$+ a\overline{y}_b\overline{y}_{aa} + (b\overline{y}_b - a\overline{y}_a)\overline{y}_{ab} - b\overline{y}_a\overline{y}_{bb} = 0 , \qquad (5.1.1)$$

and $$\overline{x}_a\overline{y}_b - \overline{x}_b\overline{y}_a = 1 . \qquad (5.1.2)$$

Since $\overline{x} = a + \varepsilon x$ and $\overline{y} = b + \varepsilon y$, (5.1.1) becomes

$$-ax_{ab} - bx_{bb} + ay_{aa} + by_{ab} = 0 \qquad (5.1.3)$$

to first order in ε, and (5.1.2) becomes

$$x_a + y_b = 0 . \qquad (5.1.4)$$

Differentiate/

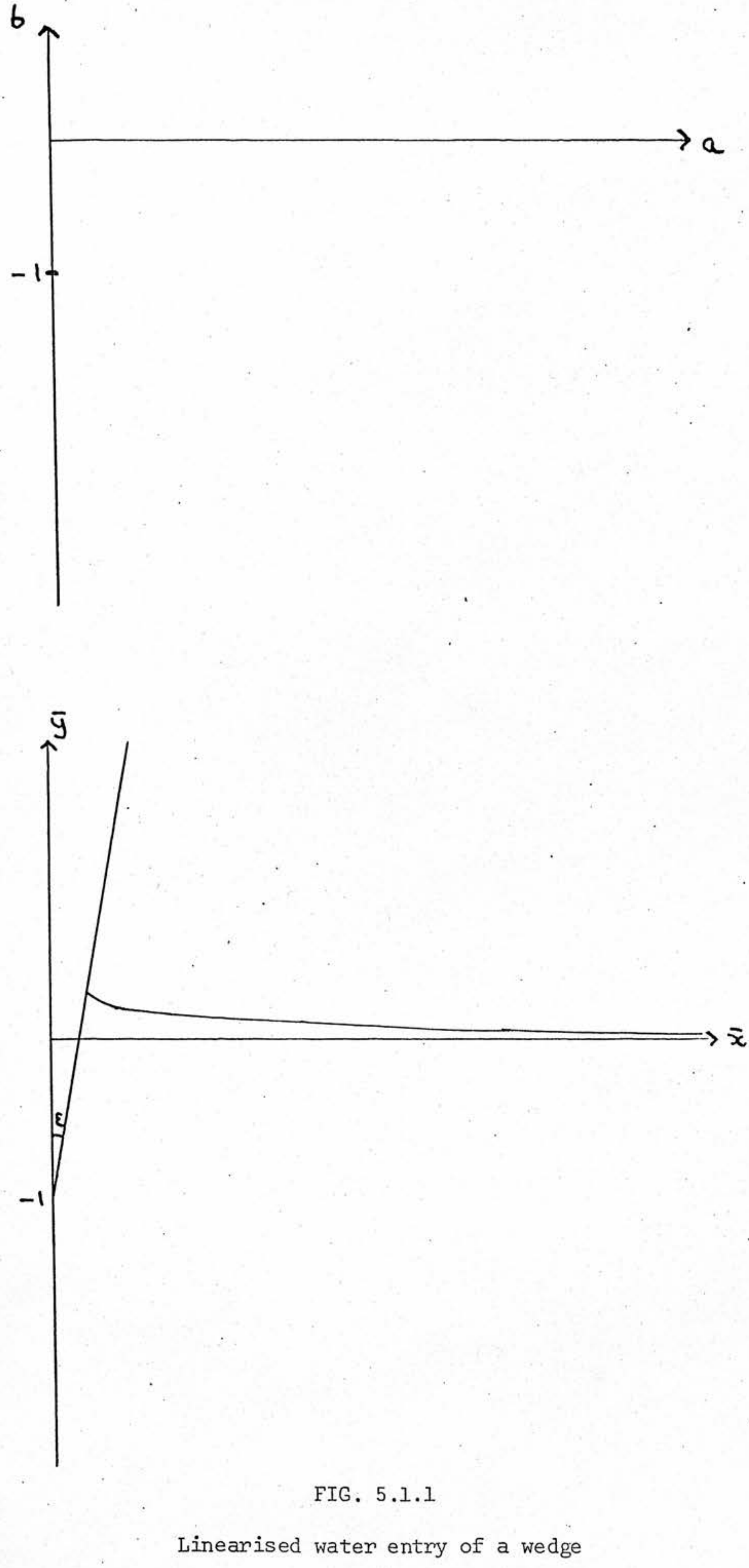


FIG. 5.1.1

Linearised water entry of a wedge

Differentiate (5.1.3) with respect to b to obtain

$$-ax_{abb} - x_{bb} - bx_{bbb} + ay_{aab} + y_{ab} + by_{abb} = 0$$

$$\Rightarrow \quad ax_{aaa} + bx_{aab} + ax_{abb} + bx_{bbb} + x_{aa} + x_{bb} = 0 \qquad (5.1.5)$$

using (5.1.4).

If this equation can be solved subject to suitable boundary conditions then y may be found from equation (5.1.4) and the pressure, if required, from the linearised versions of (3.2.7) and (3.2.8). The boundary conditions for the full non-linear problem are given by (3.4.1)-(3.4.4). In terms of the variables $\overline{x}$ and $\overline{y}$ these are

$$\overline{x}_a^{\,2} + \overline{y}_a^{\,2} = 1 \quad \text{on} \quad b = 0, \quad a \geqslant 0 \qquad (5.1.6)$$

$$\overline{x} = (\overline{y}+1)\tan\alpha \quad \text{on} \quad a = 0, \ -\ell \leqslant b \leqslant 0 \qquad (5.1.7)$$

$$\overline{x} = 0 \quad \text{on} \quad a = 0, \quad b \leqslant -\ell \qquad (5.1.8)$$

$$\overline{x} \to a \quad \text{and} \quad \overline{y} \to b \quad \text{as} \quad a^2 + b^2 \to \infty \,. \qquad (5.1.9)$$

In terms of x and y, where only quantities of first order in ε are retained, these become

$$x = 0 \quad \text{on} \quad b = 0, \ a > 0 \qquad (5.1.10)$$

$$x = b+1 \quad \text{on} \quad a = 0, \ -1 \leqslant b < 0 \qquad (5.1.11)$$

$$x = 0 \quad \text{on} \quad a = 0, \ b \leqslant -1 \qquad (5.1.12)$$

$$x \to 0 \quad \text{and} \quad y \to 0 \quad \text{as} \quad a^2 + b^2 \to \infty \,. \qquad (5.1.13)$$

To obtain (5.1.10) it is necessary to use the fact that $x \to 0$ as $a \to \infty$. The assertion that in this case $\ell = +1$ may be justified by demanding that x is continuous at $(0,-1)$.

We should now like to solve (5.1.5) subject to the boundary conditions (5.1.10)-(5.1.13).

Let/

Let $\psi(a,b) = x_{aa} + x_{bb}$. Then equation (5.1.5) may be written as

$$a\psi_a + b\psi_b + \psi = 0 \ . \tag{5.1.14}$$

It is possible to write down the general solution of this equation. Let $a = \rho \cos \chi$ and $b = \rho \sin \chi$.
Then $a\psi_a + b\psi_b = \rho\psi_\rho$ and (5.1.14) reduces to

$$\rho\psi_\rho + \psi = 0 \ .$$

This may be integrated to give $\psi = h(\chi)/\rho$ for some arbitrary function h. This function may be determined by considering the behaviour of the fluid for large ρ. If it is assumed that, at large distances from the origin, the flow is like that due to a dipole at the origin, then in this linearised theory

$$x \sim -\frac{1}{\pi} \frac{\sin 2\chi}{3\rho^2}$$

(see Appendix II for details). This is clearly a harmonic function of ρ and χ and so $h(\chi) \equiv 0$. Thus, by imposing a certain type of behaviour at infinity, we see that equation (5.1.14) has the solution $\psi \equiv 0$ and $x(a,b)$ satisfies Laplace's equation. Conditions (5.1.10)-(5.1.12) prescribe x on the boundary of the domain and so the solution for x may be written in terms of the Green's function for the quarter-plane.

Thus $\quad x(a,b) = \int_{-1}^{0} \frac{\partial G}{\partial a'}(a,b;\, 0,b')(1+b')db'$

where $G(a,b;\, a',b') = -\frac{1}{4\pi}\{\log((b-b')^2+(a-a')^2)-\log((b-b')^2+(a+a')^2)$

$$+ \log((b+b')^2+(a+a')^2)-\log((b+b')^2+(a-a')^2)\} \ .$$

Hence/

Hence $\quad x(a,b) = \frac{a}{\pi}\int_{-1}^{0}\left\{\frac{(1+b')}{(b-b')^2+a^2} - \frac{(1+b')}{(b+b')^2+a^2}\right\}db'$

$$\Rightarrow x(a,b) = -\frac{a}{2\pi}\log\frac{(1+b)^2+a^2}{(1-b)^2+a^2} + \frac{1}{\pi}\left\{(1+b)(\tan^{-1}(\frac{1+b}{a}) - \tan^{-1}\frac{b}{a})\right.$$

$$\left. - (1-b)(\tan^{-1}(\frac{1-b}{a}) + \tan^{-1}\frac{b}{a})\right\} \quad . \qquad (5.1.15)$$

It is easy to verify that this is a harmonic function and satisfies the required boundary conditions. Now $y(a,b)$ may be found from equation (5.1.4).

$$x_a(a,b) = -\frac{1}{2\pi}\log\frac{(1+b)^2+a^2}{(1-b)^2+a^2} + \frac{2}{\pi}\frac{b}{a^2+b^2} \quad .$$

Hence $\quad y(a,b) = \frac{1}{\pi}\int\left\{\frac{1}{2}\log\frac{(1+b)^2+a^2}{(1-b)^2+a^2} - \frac{2b}{a^2+b^2}\right\}db + k_1(a)$

for some arbitrary function $k_1(a)$. It is easy to show that

$$\int\log((1+b)^2+a^2)db = (1+b)\log((1+b)^2+a^2) - 2(1+b)$$
$$+ 2a\tan^{-1}((1+b)/a) \quad .$$

Thus $\quad y(a,b) = \frac{1}{2\pi}\{(1+b)\log((1+b)^2+a^2)+(1-b)\log((1-b)^2+a^2)\}$

$$+ \frac{1}{\pi}\{-2 + a\tan^{-1}(\frac{1+b}{a}) + a\tan^{-1}(\frac{1-b}{a}) - \log(a^2+b^2)\} + k_1(a).$$

It remains to define the function $k_1(a)$. Now from equations (5.1.3) and (5.1.4) it can be shown that $y(a,b)$ also satisfies Laplace's equation, provided the behaviour at infinity is like that due to a dipole, in which case $y \sim \cos 2\chi/3\pi\rho^2$ for large ρ . Since $y(a,b)$ satisfies Laplace's equation it follows that $k_1''(a) = 0$, which in turn implies that $k_1(a) = \lambda_1 a + \mu_1$, for constants λ_1 and μ_1 .

Now $\quad y(a,0) = \frac{1}{\pi}\left\{\log(1+\frac{1}{a^2}) - 2 + 2a\tan^{-1}\frac{1}{a}\right\} + \lambda_1 a + \mu_1$

$\Rightarrow \quad y(a,0) \sim \frac{1}{\pi}\{\frac{1}{3a^2} + O(\frac{1}{a^4})\} + \lambda_1 a + \mu_1$ for large a .

Since/

Since $y \sim \cos 2\chi/3\pi\rho^2$ for large ρ, it follows that

$\lambda_1 = \mu_1 = 0$ and hence that

$$y(a,b) = \frac{1}{2\pi}\{(1+b)\log((1+b)^2+a^2) + (1-b)\log((1-b)^2+a^2)$$

$$+ \frac{1}{\pi}\{-2 + a\tan^{-1}(\frac{1+b}{a}) + a\tan^{-1}(\frac{1-b}{a}) - \log(a^2+b^2)\} \quad .(5.1.16)$$

To complete this section let us calculate the pressure $\pi(a,b)$.

The linearised forms of (3.2.7) and (3.2.8) are clearly

$$a^2x_{aa} + 2ab\, x_{ab} + b^2x_{bb} = -\pi_a \qquad (5.1.17)$$

$$a^2y_{aa} + 2ab\, y_{ab} + b^2y_{bb} = -\pi_b \quad . \qquad (5.1.18)$$

Differentiation of (5.1.15) shows that

$$x_a = -\frac{1}{2\pi}\log\frac{(1+b)^2+a^2}{(1-b)^2+a^2} + \frac{2b}{\pi(a^2+b^2)}$$

$$\Rightarrow x_{aa} = -x_{bb}$$

$$= \frac{1}{\pi}\{\frac{-a}{(1+b)^2+a^2} + \frac{a}{(1-b)^2+a^2} - \frac{4ab}{(a^2+b^2)^2}\} \ .$$

Also $$x_{ab} = -\frac{1}{\pi}\{\frac{1+b}{(1+b)^2+a^2} + \frac{1-b}{(1-b)^2+a^2} - \frac{2(a^2-b^2)}{(a^2+b^2)^2}\} \ .$$

Hence $$-\pi_a(a,b) = \frac{1}{\pi}\{-\frac{a(a^2+b^2+2b)}{(1+b)^2+a^2} + \frac{a(a^2+b^2-2b)}{(1-b)^2+a^2}\} \ .$$

Integration yields

$$\pi(a,b) = \frac{1}{2\pi}\log\frac{a^2+(1-b)^2}{a^2+(1+b)^2} + k_2(b)$$

for some arbitrary function $k_2(b)$.

Now $\pi(a,b)$ must also satisfy (5.1.18) .

$$-\pi_b(a,b) = \frac{1}{\pi}\{\frac{1-b}{(1-b)^2+a^2} + \frac{1+b}{(1+b)^2+a^2}\} + k_2'(b) \ ,$$

and (5.1.18) is satisfied provided $k_2(b)$ is constant.

Since $\pi(a,0) = k_2(0)$ and the pressure is zero on the free surface, $k_2(0) = 0$.

Hence/

Hence $\quad \pi(a,b) = \dfrac{1}{2\pi} \log \dfrac{a^2+(1-b)^2}{a^2+(1+b)^2} \quad .$ (5.1.19)

This demonstrates that at least in the linearised theory the pressure is positive everywhere within the fluid since $(a^2+(1-b)^2)/(a^2+(1+b)^2) > 1$ for $b < 0$ and $a \geq 0$. In particular this holds on the wedge face (where $-1 < b < 0$ and $a = 0$), and so we expect the free surface to be convex. On the free surface, since $x = 0$ there, $\bar{x} = a$, and

$$\bar{y} = \frac{\varepsilon}{\pi}\left\{\log(1 + \frac{1}{\bar{x}^2}) - 2 + 2\bar{x}\tan^{-1}\frac{1}{\bar{x}}\right\} . \qquad (5.1.20)$$

This agrees with the form of the free surface which may be obtained from a linearised approach using Eulerian coordinates, as was shown by Mackie (1962).

Although the quantities x_a, x_b, y_a, y_b may be found simply by differentiating (5.1.15) and (5.1.16), they may also be calculated directly from a linearised version of the first-order system (3.5.3) with boundary conditions supplied by (3.7.1)-(3.7.4). The result obtained in each case is

$$x_a = -y_b = \frac{1}{2\pi}\log\frac{a^2+(1-b)^2}{a^2+(1+b)^2} + \frac{2}{\pi}\,\frac{b}{a^2+b^2} \qquad (5.1.21)$$

and $$x_b = y_a = \frac{1}{\pi}\{\tan^{-1}(\frac{1+b}{a}) + \tan^{-1}(\frac{1-b}{a})\} - \frac{2}{\pi}\,\frac{a}{a^2+b^2} \quad . \qquad (5.1.22)$$

(5.2) <u>Linearised theory for A, B, ϕ and ψ.</u>

It is a simple matter to produce linearised forms of the functions $A(\rho,\theta)$ and $B(\rho,\theta)$ and the angles $\phi(\rho,\theta)$ and $\psi(\rho,\theta)$, since linearised forms are known for $x(\rho,\theta)$ and $y(\rho,\theta)$. From (5.1.15) we can write

$\bar{x}$ /

$$\bar{x}(\rho,\theta) = \rho \cos\theta + \frac{\varepsilon}{\pi}\{ -\frac{\rho\cos\theta}{2}\log(\frac{\rho^2+2\rho\sin\theta+1}{\rho^2-2\rho\sin\theta+1})$$

$$+ (1+\rho\sin\theta)\{\tan^{-1}(\frac{1+\rho\sin\theta}{\rho\cos\theta})-\theta\}-(1-\rho\sin\theta)\{\tan^{-1}(\frac{1-\rho\sin\theta}{\rho\cos\theta})\}\}. \tag{5.2.1}$$

Also, from (5.1.16) it follows that

$$\bar{y}(\rho,\theta) = \rho\sin\theta + \frac{\varepsilon}{\pi}\{\tfrac{1}{2}(1+\rho\sin\theta)\log(1+2\rho\sin\theta+\rho^2)$$

$$+ \tfrac{1}{2}(1-\rho\sin\theta)\log(1-2\rho\sin\theta+\rho^2) - 2 - 2\log\rho$$

$$+ \rho\cos\theta(\tan^{-1}(\frac{1+\rho\sin\theta}{\rho\cos\theta}) + \tan^{-1}(\frac{1-\rho\sin\theta}{\rho\cos\theta}))\}. \tag{5.2.2}$$

Now $A(\rho,\theta)$ is defined by (3.9.7) and (3.9.8). In this present case let us write $\bar{A}(\rho,\theta)$ for $A(\rho,\theta)$ so that

$$\bar{A}_\rho = \rho(\bar{x}_\rho{}^2 + \bar{y}_\rho{}^2)$$

$$\bar{A}_\theta = \rho(\bar{x}_\rho\bar{x}_\theta + \bar{y}_\rho\bar{y}_\theta).$$

Clearly $\bar{A} = \rho^2/2$ when $\varepsilon = 0$. Let $\bar{A} = \rho^2/2 + \varepsilon A$.

Then $A_\rho = 2\rho\,(x_\rho\cos\theta + y_\rho\sin\theta)$

and $A_\theta = \rho(x_\theta\cos\theta + y_\theta\sin\theta - \rho x_\rho\sin\theta + \rho y_\rho\cos\theta)$.

Hence, from (5.2.1) and (5.2.2) we have

$$A_\rho = \frac{1}{\pi}\{-\rho\cos 2\theta\ \log(\frac{1+2\rho\sin\theta+\rho^2}{1-2\rho\sin\theta+\rho^2})+2\rho\sin 2\theta\{\tan^{-1}(\frac{1+\rho\sin\theta}{\rho\cos\theta})+\tan^{-1}(\frac{1-\rho\sin\theta}{\rho\cos\theta})\} - 4\sin\theta\}.$$

$$A_\theta = \frac{1}{\pi}\{\rho^2\sin 2\theta\ \log(\frac{1+2\rho\sin\theta+\rho^2}{1-2\rho\sin\theta+\rho^2})+2\rho^2\cos 2\theta\{\tan^{-1}(\frac{1+\rho\sin\theta}{\rho\cos\theta})+\tan^{-1}(\frac{1-\rho\sin\theta}{\rho\cos\theta})\} - 4\rho\cos\theta.$$

Integration of A_ρ with respect to ρ gives

$$\pi A = -\frac{\rho^2\cos 2\theta}{2}\log(\frac{1+2\rho\sin\theta+\rho^2}{1-2\rho\sin\theta+\rho^2})+\rho^2\sin 2\theta\{\tan^{-1}(\frac{1+\rho\sin\theta}{\rho\cos\theta})+\tan^{-1}(\frac{1-\rho\sin\theta}{\rho\cos\theta})\}$$

$$-\tfrac{1}{2}\log(\frac{1+2\rho\sin\theta+\rho^2}{1-2\rho\sin\theta+\rho^2}) - 2\rho\sin\theta + g(\theta)$$

for some function $g(\theta)$. If we differentiate this expression with respect to θ we see that $g(\theta)$ is constant and since

A/

$A \to 0$ as $\rho \to \infty$,

$$A(\rho,\theta)= \frac{1}{\pi}\{- \frac{\rho^2\cos 2\theta}{2}\log(\frac{1+2\rho\sin\theta+\rho^2}{1-2\rho\sin\theta+\rho^2}) + \rho^2\sin 2\theta\{\tan^{-1}(\frac{1+\rho\sin\theta}{\rho\cos\theta})$$

$$+ \tan^{-1}(\frac{1-\rho\sin\theta}{\rho\cos\theta})\} - \tfrac{1}{2}\log(\frac{1+2\rho\sin\theta+\rho^2}{1-2\rho\sin\theta+\rho^2}) - 2\rho\sin\theta\}. \qquad (5.2.3)$$

Define $\overline{B}(\rho,\theta)$ by analogy with (3.9.18) and (3.9.21) as

$$\overline{B}_\rho = \frac{\overline{A}_\rho}{\rho}\frac{\partial}{\partial\rho}(\frac{\overline{A}_\theta}{\rho\overline{A}_\rho})$$

$$\overline{B}_\theta + 2 = \frac{3\rho}{\overline{A}_\rho} - \frac{\rho^2\overline{A}_{\rho\rho}}{\overline{A}_\rho{}^2} + \frac{\overline{A}_\theta}{\rho}\frac{\partial}{\partial\rho}(\frac{\overline{A}_\theta}{\rho\overline{A}_\rho}) .$$

Clearly $\overline{B} = 0$ when $\varepsilon = 0$. Let $\overline{B} = \varepsilon B$.

Then $$B_\rho = \frac{\partial}{\partial\rho}(\frac{A_\theta}{\rho^2})$$

$$\Rightarrow B = \frac{A_\theta}{\rho^2} + h(\theta)$$

for some function $h(\theta)$. We know from (3.9.31) and (3.9.32) that for large ρ

$$\overline{A} \sim \frac{\rho^2}{2} - \frac{4\varepsilon}{3\pi}\frac{\sin\theta}{\rho}$$

and $$\overline{B} \sim -\frac{4\varepsilon}{3\pi}\frac{\cos\theta}{\rho^3}$$

since $m = \varepsilon/\pi$ for the linearised problem. Thus $h(\theta) = 0$ and $B = A_\theta/\rho^2$.

$$B(\rho,\theta)= \frac{1}{\pi}\{\sin 2\theta\log(\frac{1+2\rho\sin\theta+\rho^2}{1-2\rho\sin\theta+\rho^2}) + 2\cos 2\theta\{\tan^{-1}(\frac{1+\rho\sin\theta}{\rho\cos\theta})$$

$$+ \tan^{-1}(\frac{1-\rho\sin\theta}{\rho\cos\theta}) - \frac{4\cos\theta}{\rho}\} . \qquad (5.2.4)$$

On the boundary $\theta = 0$, we have $\overline{A}_\rho = \rho$ and $\overline{B} = \overline{A}_\theta/\rho^2 = (4\varepsilon/\pi)(\tan^{-1}(1/\rho) - 1/\rho)$. These are consistent with the boundary conditions (3.9.27). In the linearised problem the relation $\overline{B} = \overline{A}_\theta/\rho^2$ holds throughout the region of validity of the solution and not just on the free surface.

On/

On $\theta = -\pi/2, \rho > 1$, $\overline{A}_\rho = \rho + \frac{2\varepsilon}{\pi}\{\rho \log \frac{\rho-1}{\rho+1} + 2\}$

$\overline{A}_\theta = \overline{B} = 0$.

On $\theta = -\pi/2, \rho < 1$, $\overline{A}_\rho = \rho + \frac{2\varepsilon}{\pi}\{\rho \log \frac{1-\rho}{1+\rho} + 2\}$

$\overline{A}_\theta = \rho^2\,\overline{B} = -2\varepsilon$.

Once again these are consistent with the boundary conditions (3.9.28) and (3.9.29). In this case
$\overline{A}_\theta - 2\varepsilon\rho\overline{A}_\rho = O(\varepsilon^2)$ on $\theta = -\pi/2, \rho < 1$ and the constant K is equal to -2ε .

If we define ϕ and ψ by the equations

$\phi = \overline{B}/2$ and $\cot(\psi-\phi) = \overline{A}_\theta/\rho^2$, then

$\phi = \varepsilon B/2$ and $\psi = \pi/2 - \varepsilon B/2$. Hence

$$\phi= \frac{\varepsilon}{\pi}\left\{- \frac{2\cos\theta}{\rho} + \frac{\sin2\theta}{2}\log\left(\frac{1+2\rho\sin\theta+\rho^2}{1-2\rho\sin\theta+\rho^2}\right) + \cos2\theta\left\{\tan^{-1}\left(\frac{1+\rho\sin\theta}{\rho\cos\theta}\right)+ \tan^{-1}\left(\frac{1-\rho\sin\theta}{\rho\cos\theta}\right)\right\}\right.$$

$$\psi= \frac{\pi}{2} - \frac{\varepsilon}{\pi}\left\{\frac{2\cos\theta}{\rho} + \frac{\sin2\theta}{2}\log\left(\frac{1+2\rho\sin\theta+\rho^2}{1-2\rho\sin\theta+\rho^2}\right)+\cos2\theta\left\{\tan^{-1}\left(\frac{1+\rho\sin\theta}{\rho\cos\theta}\right)+ \tan^{-1}\left(\frac{1-\rho\sin\theta}{\rho\cos\theta}\right)\right\}\right.$$

In this case the relation $\psi = \pi/2 - \phi$ holds throughout the region of validity of the solution and not just for large values of ρ .
On the free surface, as ρ decreases, ϕ decreases and ψ increases. There is no maximum of ψ and so no point where $\psi = \pi/2$ for finite ρ. Also the convected circles meet the wedge face at a constant angle $(\pi/2 + 2\varepsilon)$ since for $0 < \rho < 1$, $\psi(\rho,-\pi/2) - \phi(\rho,-\pi/2) = \pi/2 + 2\varepsilon$. The area of greatest interest is a small neighbourhood of the contact point which is where the linearised theory unfortunately breaks down.

(5.3) Connection between Eulerian and Lagrangian coordinates.

It is to be expected that, in some sense, an Eulerian and a Lagrangian description of a particular flow are equivalent. Ideally one would hope to be able to deduce one form of the solution once

once the other is known. Let us suppose that we know a potential function $\Phi(X,Y,t)$ which describes some flow regime. If, in Lagrangian variables, $U(A,B,t)$ and $V(A,B,t)$ are velocity components in the X- and Y-directions of a particle which was initially at (A,B), and has subsequently moved to (X,Y), then

$$X = A + \int_0^t U(A,B,t)dt \quad \text{and} \quad Y = B + \int_0^t V(A,B,t)\,dt \tag{5.3.1}$$

or alternatively $U(A,B,t) = X_t(A,B,t)$ and $V(A,B,t) = Y_t(A,B,t)$. Since by $U(A,B,t)$ we mean the velocity in the X-direction at (X,Y) of a particle which was initially at (A,B) we have

$$\frac{\partial\Phi}{\partial X}(X,Y,t) \equiv \frac{\partial X}{\partial t}(A,B,t)$$

$$\Rightarrow \frac{\partial\Phi}{\partial X}\left(A + \int_0^t U(A,B,t)dt, B + \int_0^t V(A,B,t)dt, t\right) \equiv \frac{\partial X}{\partial t}(A,B,t). \tag{5.3.2}$$

Similarly

$$\frac{\partial\Phi}{\partial Y}\left(A+\int_0^t U(A,B,t)dt,\ B+\int_0^t V(A,B,t)dt, t\right) \equiv \frac{\partial Y}{\partial t}(A,B,t)\ . \tag{5.3.3}$$

These formulae give the required formal connection between the Eulerian and Lagrangian variables although in practice they may be very difficult to handle. If a linearised theory is available considerable simplification takes place. If we take $\overline{X} = A + \varepsilon X$ and $\overline{Y} = B + \varepsilon Y$ as the coordinates at time t of a particle initially at (A,B) and if the velocity potential is taken to be $\overline{\Phi} = \varepsilon\Phi$, then $U(A,B,t)$ and $V(A,B,t)$ are $O(\varepsilon)$ and expanding $\partial\Phi/\partial X$ and $\partial\Phi/\partial Y$ about (A,B,t) and retaining only first order terms in ε, we find that

$$\frac{\partial\Phi}{\partial X}(A,B,t) = \frac{\partial X}{\partial t}(A,B,t) \tag{5.3.4}$$

$$\frac{\partial\Phi}{\partial Y}(A,B,t) = \frac{\partial Y}{\partial t}(A,B,t)\ . \tag{5.3.5}$$

These/

These relations can easily be verified in the case of the water-entry problem. From (5.1.15),

$$x(a,b) = -\frac{a}{2\pi}\log\frac{(1+b)^2+a^2}{(1-b)^2+a^2} + \frac{1}{\pi}\{(1+b)(\tan^{-1}(\frac{1+b}{a}) - \tan^{-1}\frac{b}{a})$$

$$- (1-b)(\tan^{-1}(\frac{1-b}{a}) + \tan^{-1}\frac{b}{a})\}$$

$$\Rightarrow \quad X(A,B,t) = V_o t \; x(\frac{A}{V_o t}, \frac{B}{V_o t})$$

$$= -\frac{A}{2\pi}\log(\frac{(V_o t+B)^2+A^2}{(V_o t-B)^2+A^2}) + \frac{1}{\pi}\{(V_o t+B)(\tan^{-1}(\frac{V_o t+B}{A}) - \tan^{-1}\frac{B}{A})$$

$$- (V_o t-B)(\tan^{-1}(\frac{V_o t-B}{A}) + \tan^{-1}\frac{B}{A})\}$$

$$\Rightarrow X_t(A,B,t) = -\frac{V_o}{\pi}\{\tan^{-1}(\frac{V_o t+B}{A}) - \tan^{-1}(\frac{V_o t-B}{A}) + 2\tan^{-1}\frac{B}{A}\} \quad .(5.3.6)$$

From (5.1.16)

$$y(a,b) = \frac{1}{2\pi}\{(1+b)\log((1+b)^2+a^2) + (1-b)\log((1-b)^2+a^2)\}$$

$$+ \frac{1}{\pi}\{-2 + a\tan^{-1}(\frac{1+b}{a}) + a\tan^{-1}(\frac{1-b}{a}) - \log(a^2+b^2)\}$$

$$\Rightarrow Y(A,B,t) = V_o t \; y(\frac{A}{V_o t}, \frac{B}{V_o t})$$

$$= \frac{1}{2\pi}\{(V_o t+B)\log\frac{(V_o t+B)^2+A^2}{V_o^2 t^2} + (V_o t-B)\log\frac{(V_o t+B)^2+A^2}{V_o^2 t^2}\}$$

$$+ \frac{1}{\pi}\{-2V_o t + A\tan^{-1}(\frac{V_o t+B}{A}) + A\tan^{-1}(\frac{V_o t-B}{A}) - V_o t\log\frac{A^2+B^2}{V_o^2 t^2}\}$$

$$\Rightarrow Y_t(A,B,t) = \frac{V_o}{\pi}\{\tfrac{1}{2}\log(V_o t+B)^2+A^2) + \tfrac{1}{2}\log(V_o t-B)^2+A^2) - \log(A^2+B^2)\}$$

(5.3.7)

$$\text{Now } \phi(\bar{x},\bar{y},t) = \frac{1}{2\pi}\int_0^1 \log\frac{\bar{x}^2+(\bar{y}+\theta)^2}{\bar{x}^2+(\bar{y}-\theta)^2}\,d\theta \quad \text{(from (2.4.7))}$$

$$\Rightarrow \quad \Phi(\bar{X},\bar{Y},t) = V_o^2 t\,\phi(\bar{X}/V_o t, \bar{Y}/V_o t)$$

$$= \frac{V_o^2 t}{2\pi}\int_0^1 \log\frac{\bar{X}^2+(\bar{Y}+V_o t\theta)^2}{\bar{X}^2+(\bar{Y}-V_o t\theta)^2}\,d\theta$$

=>/

$$= \frac{\partial\Phi}{\partial\overline{X}} = \frac{V_o}{\pi}\{\tan^{-1}\left(\frac{V_o t + \overline{Y}}{\overline{X}}\right) - \tan^{-1}\frac{V_o t - \overline{Y}}{\overline{X}} - 2\tan^{-1}\frac{\overline{Y}}{\overline{X}}\} \quad (5.3.8)$$

and
$$\frac{\partial\Phi}{\partial\overline{Y}} = \frac{V_o}{\pi}\{\tfrac{1}{2}\log((\overline{X}^2+(\overline{Y}+V_o t)^2)+ \tfrac{1}{2}\log(\overline{X}^2+(\overline{Y}-V_o t)^2)$$
$$- \log(\overline{X}^2+ \overline{Y}^2)\ \} . \quad (5.3.9)$$

But $\overline{X} = A + \varepsilon X$ and $\overline{Y} = B + \varepsilon Y$ and so substitution in (5.3.8) and (5.3.9) shows that the velocities agree to the required order of accuracy. In fact what has been shown is

$$\left[\frac{\partial\Phi}{\partial\overline{X}}\right]_{\overline{X}=A+\varepsilon X,\overline{Y}=B+\varepsilon Y} - \frac{\partial X}{\partial t}(A,B,t) = O(\varepsilon)$$

and
$$\left[\frac{\partial\Phi}{\partial\overline{Y}}\right]_{\overline{X}=A+\varepsilon X,\overline{Y}=B+\varepsilon Y} - \frac{\partial Y}{\partial t}(A,B,t) = O(\varepsilon) .$$

Both versions of the linearised theory break down sufficiently near the tip of the wedge and the contact point, but the above analysis shows that they are equivalent to the anticipated order of approximation.

CHAPTER 6.

(6.1) Equations of motion for the cone water-entry problem.

Let us use the same notation as for the wedge problem except that the y-axis is now the axis of symmetry and x is a radial coordinate (see Figure 6.1.1). The equations which express a balance of forces are identical to those in the two-dimensional wedge problem and are (3.2.1) and (3.2.2) with $\underline{F} = 0$ and it follows that vorticity is conserved according to equation (3.2.6). In terms of the similarity variables the equations of motion are then identical to (3.2.7)-(3.2.9) and are

$$(a^2x_{aa}+ 2ab\, x_{ab}+ b^2x_{bb})x_a+(a^2y_{aa}+ 2ab\, y_{ab}+ b^2y_{bb})y_a = -\pi_a \quad (6.1.1)$$

$$(a^2x_{aa}+ 2ab\, x_{ab}+ b^2x_{bb})x_b+(a^2y_{aa}+ 2ab\, y_{ab}+ b^2y_{bb})y_b = -\pi_b \quad (6.1.2)$$

$$ax_bx_{aa}+(bx_b-ax_a)x_{ab}- bx_ax_{bb}$$
$$+ ay_by_{aa}+(by_b- ay_a)y_{ab} - by_ay_{bb} = 0 . \quad (6.1.3)$$

The continuity equation is not the same however. In the axisymmetric case when we consider a small element of fluid, what is really meant is an entire annulus of fluid of small cross-section (see Figure 6.1.2). Suppose the small element of fluid is situated, in the meridian plane, with corners at $t = 0$ at (A,B), $(A+\delta A,B)$, $(A,B+\delta B)$ and $(A+\delta A,B+\delta B)$. At $t = 0$ the volume of this element is $2\pi A\ \delta A\ \delta B$. At later time it

is/

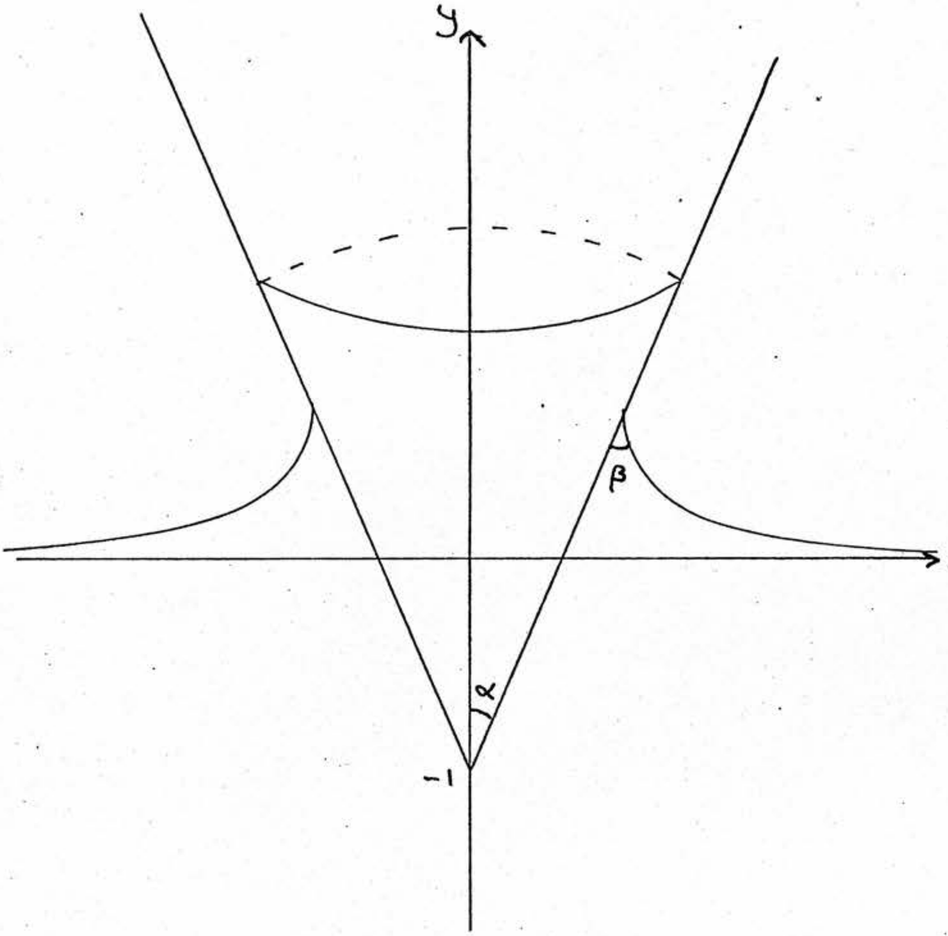


FIG. 6.1.1

Water entry of a cone

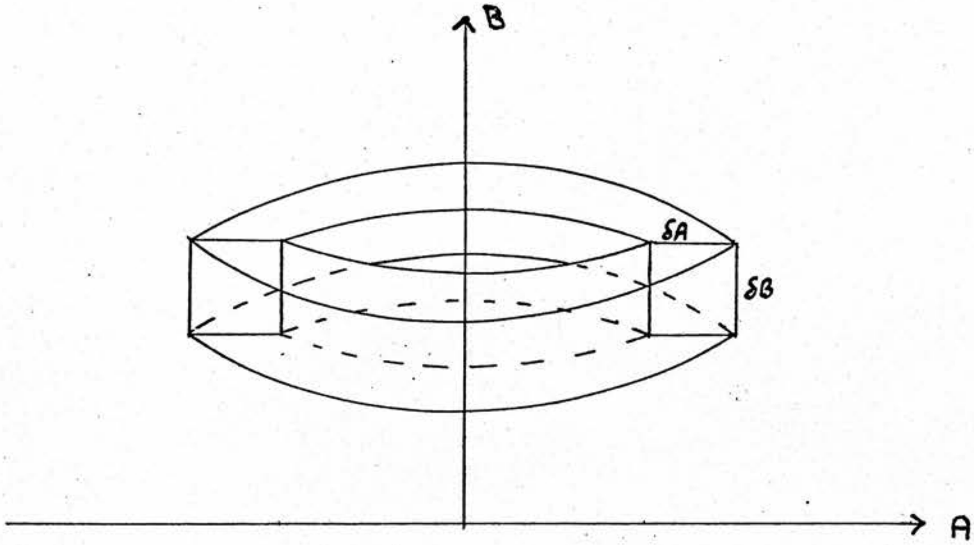


FIG. 6.1.2

A small element of fluid in an axisymmetric problem is $2\pi X(X_A Y_B - X_B Y_A)\delta A \delta B$ + higher order terms. Since the fluid is incompressible the continuity equation is

$$X(X_A Y_B - X_B Y_A) = A \quad . \tag{6.1.4}$$

Differentiation with respect to t produces the alternative form

$$X_{At}Y_B + X_A Y_{Bt} - X_{Bt}Y_A - X_B Y_{At} = -\frac{X_t}{X}(X_A Y_B - X_B Y_A) \; . \tag{6.1.5}$$

In the similarity variables (6.1.4) becomes

$$x(x_a y_b - x_b y_a) = a \tag{6.1.6}$$

and (6.1.5) becomes

$$ay_b x_{aa} + (by_b - ay_a)x_{ab} - by_a x_{bb} - ax_b y_{aa} - (bx_b - ax_a)y_{ab} + bx_a y_{bb}$$
$$= \frac{a}{x^2}(x - ax_a - bx_b) \; . \tag{6.1.7}$$

As in the two-dimensional case the equations (6.1.3) and (6.1.7) may be written as a first-order system.

As before, let $p = x_a$, $q = x_b$, $r = y_a$ and $s = y_b$.

Thus (6.1.3) and (6.1.7) become respectively

$$aqp_a - apq_a + asr_a - ars_a + bqp_b - bpq_b + bsr_b - brs_b = 0 \; .$$

$$asp_a - arq_a - aqr_a + aps_a + bsp_b - brq_b - bqr_b + bps_b$$
$$= (ps - qr) - \frac{1}{a}(ap + bq)(ps - qr)^2 \; .$$

There are also the consistency equations

$$q_a - p_b = 0$$
$$s_a - r_b = 0 \; .$$

In matrix notation these four equations may be written as

$$A\underline{u}_a + B\underline{u}_b = \underline{f}$$

where/

where $A = \begin{bmatrix} aq & -ap & as & -ar \\ as & -ar & -aq & ap \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}$, $B = \begin{bmatrix} bq & -bp & bs & -br \\ bs & -br & -bq & bp \\ -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \end{bmatrix}$,

$\underline{u} = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}$ and $\underline{f} = \begin{bmatrix} \cdot \\ ps-qr-(ap+bq)(ps-qr)^2/a \\ \cdot \\ \cdot \end{bmatrix}$.

This system is very similar to that obtained for the two-dimensional problem. The matrices A and B and the vector $\underline{u}$ are unchanged but now there is a non-zero right hand side $\underline{f}$. Many of the properties of the equations are unchanged by the presence of this vector $\underline{f}$ although the consequent lack of homogeneity does introduce new complications. In particular the real characteristics are still given by $da/db = a/b$ and on the lines $b = ma$ the relation $qdp - pdq + sdr - rds = 0$ holds.

(6.2) Boundary conditions.

As for the wedge problem there are three parts of the boundary to be considered: the free surface, the wetted cone face and the axis of symmetry. If we assume again that all the particles initially on the free surface remain there throughout the motion, then the theorem of conservation of arc-length holds along any curve on the free surface which lies in the meridian plane. It is clear that the arguments of (3.3) apply without alteration since equations (6.1.1) and (6.1.2) are identical to (3.2.7) and (3.2.8).

The/

The first boundary condition is then -

On $b = 0, \quad {x_a}^2 + {y_a}^2 = 1$. (6.2.1)

On the boundary $a = 0$ there are two conditions -

For $a = 0, \quad -\ell \leqslant b \leqslant 0, \quad x = (y+1)\tan\alpha$ (6.2.2)

and for $a = 0, \quad b \leqslant -\ell, \quad x = 0$. (6.2.3)

The constant ℓ has the same meaning as in the wedge problem. A particle initially at $(0,B)$ will reach the tip of the cone in time $t_B = -B/\ell V_o$ (see section 3.4). At infinity the motion dies away and so $x \to a$ and $y \to b$ as $a^2 + b^2 \to \infty$. If it is assumed that the behaviour at infinity is like that due to a dipole then it can be shown that, for large ρ,

$$\begin{aligned} x &\sim \rho\cos\theta - \frac{3\,m\sin\theta\cos\theta}{\rho^3} \\ y &\sim \rho\sin\theta + \frac{m(1 - 3\sin^2\theta)}{\rho^3} \quad , \end{aligned} \qquad (6.2.4)$$

where $a = \rho\cos\theta$, $b = \rho\sin\theta$ and m is a constant related to the strength of the dipole (see Appendix II for details.)

In order to check that it is plausible to postulate that particles on the axis of symmetry can move up the cone a local analysis near the tip of the cone can be carried out. Let $X = R\sin\theta$ and $Y = -R\cos\theta - V_o t$ as in Figure 6.2.1. Assume that near the tip of the cone the velocity potential $\Phi(X,Y,t)$ takes the form

$$\Phi = -V_oY + C(t)\,R^n\,P_n(\cos\theta) + o(R^n) \,, \qquad (6.2.5)$$

where $C(t)$ is an unknown function of t and n is some number, not necessarily an integer. Clearly $n > 1$ since the/

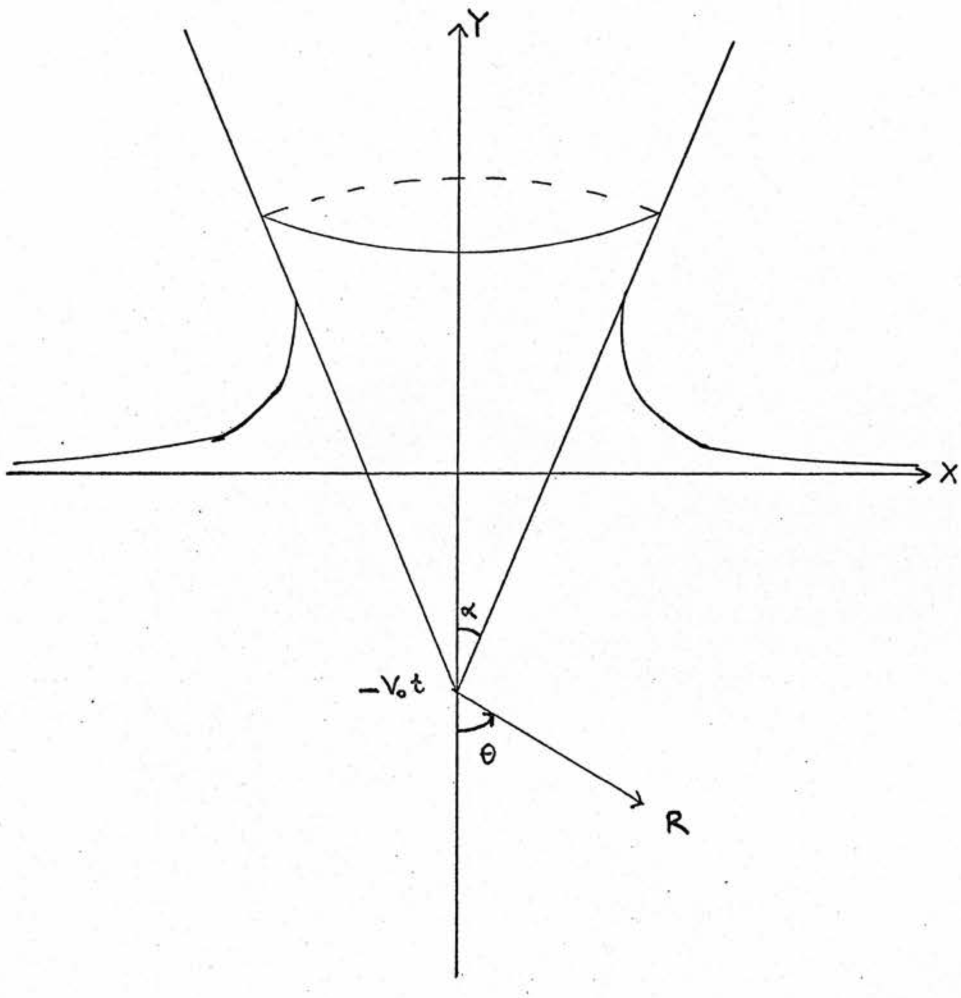


FIG. 6.2.1

Water entry of a cone

the velocity at the tip of the cone must be finite and of magnitude V_o. In what follows we shall consider the effects of a velocity potential of the form $-V_oY + C(t)\, R^n\, P_n(\cos\theta)$ on the basis that this is an approximation to the flow near the tip of the cone. Because of the similarity we can write $C(t) = C\, V_o^{2-n}\, t^{1-n}$, and then

$$\Phi = V_o(R\cos\theta + V_ot) + C\, V_o^{2-n}\, t^{1-n}\, R^n\, P_n(\cos\theta)\ .$$

We should like to consider the behaviour of particles for which $A = 0$ at times close to $t_B = -B/\ell V_o$.

Now $\dfrac{\partial\Phi}{\partial R} = V_o\cos\theta + n\, C(t)\, R^{n-1}\, P_n(\cos\theta)$

and $\dfrac{1}{R}\dfrac{\partial\Phi}{\partial\theta} = -V_o\sin\theta - \sin\theta\, C(t)R^{n-1}\, P_n{}'(\cos\theta)$.

The symmetry condition $\partial\Phi/\partial\theta = 0$ on $\theta = 0$ is automatically satisfied by the choice of this type of harmonic function. We must also have

$$\frac{1}{R}\frac{\partial\Phi}{\partial\theta} = -V_o\sin\alpha \quad \text{on} \quad \theta = \pi-\alpha\ .$$

$$\Rightarrow \quad P_n{}'(-\cos\alpha) = 0\ .$$

This is not sufficient to determine n since there are obviously many possible solutions for $n > 1$.

Now $\dfrac{\partial\Phi}{\partial X} = C(t)\, R^{n-2}\{nXP_n(\cos\theta) + \sin\theta(Y+V_ot)P_n{}'(\cos\theta)\}$

and $\dfrac{\partial\Phi}{\partial Y} = -V_o + C(t)R^{n-2}\,\{n(Y+V_ot)P_n(\cos\theta) - X\sin\theta\, P_n{}'(\cos\theta)\}$.

To find the particle paths we have to solve

$$\frac{dX}{dt} = \frac{\partial\Phi}{\partial X} \quad \text{and} \quad \frac{dY}{dt} = \frac{\partial\Phi}{\partial Y}$$

subject to $X = 0$ and $Y = -V_ot_B$ at $t = t_B$.

On/

On $\theta = 0$ we have simply

$$\frac{dX}{dt} = 0$$

$$\frac{dY}{dt} = - V_o - n\, C(t) \left(-(Y+V_o t)\right)^{n-1} .$$

The first of these gives $X = 0$, since $X = 0$ at $t = t_B$ and this holds for $t \leqslant t_B$. The second gives

$$\frac{dR}{dt} = n\, C\, V_o^{2-n} t^{1-n} R^{n-1} \quad \text{since } R = -(Y+V_o t) .$$

$$\Rightarrow R^{2-n} = n\, C\, V_o^{2-n}(t^{2-n} - t_B^{2-n}) \quad \text{since } R = 0 \text{ at } t = t_B .$$

In order that $R = 0$ is attained in a finite time, we must have $n < 2$. Thus n satisfies

$$P_n'(- \cos \alpha) = 0 \quad \text{and} \quad 1 < n < 2 . \tag{6.2.6}$$

Consultation of tables shows that there is a unique root of $P_n'(- \cos \alpha) = 0$ with $1 < n < 2$ and $0 < \alpha < \pi/2$, and so it is possible for particles on the axis of symmetry to move up the cone. This is similar to the situation which occurs in the wedge problem, as is detailed in Appendix I.

For $t < t_B$ we have

$$R^{2-n} = n\, C\, V_o^{2-n}(t^{2-n} - t_B^{2-n})$$

and in the similarity variables this gives, for $b < -\ell$,

$$x = 0$$
$$y = -1 - (-n\, C)^{1/(2-n)}\left(-1 +(-b/\ell)^{2-n}\right)^{1/(2-n)} . \tag{6.2.7}$$

For $t > t_B$, $\theta = \pi-\alpha$ and

$$\frac{dX}{dt} = n\, C(t)\, R^{n-1} \sin \alpha\, P_n(-\cos \alpha)$$

$$\frac{dY}{dt} = -V_o + n\, C(t)\, R^{n-1} \cos\alpha\, P_n(-\cos \alpha) .$$

These/

These can be integrated to give, for $b > -\ell$,

$$\left.\begin{aligned} x &= \sin\alpha\left(n\,C\,P_n(-\cos\alpha)\right)^{1/(2-n)}\{1-(-b/\ell)^{2-n}\}^{1/(2-n)} \\ y &= -1+\cos\alpha\left(nCP_n(-\cos\alpha)\right)^{1/(2-n)}\{1-(-b/\ell)^{2-n}\}^{1/(2-n)} \end{aligned}\right\} \quad (6.2.8)$$

Differentiation of (6.2.7) and (6.2.8) shows that not only x and y are continuous at $b = -\ell$ but also that q and s are continuous and equal to zero at that point.

(6.3) The contact angle.

In this axisymmetric cone problem there are no theorems concerning the size of the contact angle, since in the two-dimensional case these follow from various complex variable methods which are now unavailable. The theorem concerning the convexity of the free surface still holds in the sense that, in the meridian plane, the free surface curve is convex to the fluid provided the pressure is non-negative on the cone face. If this is the case then the inequality $\alpha+\beta \leq \pi/2$ can be deduced from the convexity of the free surface, the conservation of arc-length on it and the hypothesis that particles initially on the free surface remain there (cf. section 2.4).

If the slope of the free surface is $\tan\phi$ then it follows as in (3.6.4) and (3.6.5) that

$$r = \sin\phi \quad \text{and} \quad p = \cos\phi .$$

Now on $a = 0$, $-\ell < b < 0$, from (6.2.2) $q = s\tan\alpha$.
From the continuity equation (6.1.6) applied on $a = 0$,
$ps - qr = 0$, provided $x \neq 0$ and clearly $x \neq 0$ for $-\ell < b < 0$.
Since/

Since $ps - qr = 0$, $p = r \tan \alpha$ on $a = 0$, $-\ell < b < 0$.

As a tends to zero along the free surface $\phi \to -\left(\pi/2-(\beta-\alpha)\right)$.

Hence as $a \to 0$, $r \to \mp\cos(\beta-\alpha)$ and $p \to \sin(\beta-\alpha)$.

If p/r is continuous at the origin then

$$\tan \alpha = \tan(\alpha-\beta)$$

$$\Rightarrow \quad \beta = 0, \text{ since } \alpha+\beta \lesssim \pi/2 . \tag{6.3.1}$$

This is an interesting result although it depends on the hypothesis that p/r is continuous at the origin. Tayler (1972) has shown that for the cone-entry problem the only possible contact angles apart from zero are $\pi/2$ and π. If $\pi/2$ and π are rejected on the grounds of convexity and conservation of arc-length, then we are left with the sole possibility of $\beta = 0$ which is confirmed by the above result. Of course, similar reservations apply to this problem as to the two-dimensional problem. Apart from the assumption of continuity at the origin, there are also the claims that the particle initially at the origin is subsequently sited at the contact point and that the pressure is non-negative on the cone face, neither of which has been rigourously established. Finally, it is interesting to note that Shiffman and Spencer (1951) made without comment the assumption that the contact is tangential. Since that time a considerable amount of work has been done to show that tangential contact is impossible for the wedge problem, and this might lead one to suppose that this is the case for the cone problem also. It would be ironic if Shiffman and Spencer were correct after all.

(6.4) <u>The functions A and B and the angles ϕ and ψ.</u>

In polar coordinates the equations of motion (6.1.3) and (6.1.6) are

$$x_\theta x_{\rho\rho} - x_\rho x_{\rho\theta} + \frac{x_\rho x_\theta}{\rho} + y_\theta y_{\rho\rho} - y_\rho y_{\rho\theta} + \frac{y_\rho y_\theta}{\rho} = 0 \qquad (6.4.1)$$

$$x(x_\rho y_\theta - x_\theta y_\rho) = \rho^2 \cos\theta \qquad (6.4.2)$$

where $a = \rho\cos\theta$ and $b = \rho\sin\theta$. The first of these may be written as

$$\frac{\partial}{\partial\rho}(\rho(x_\rho x_\theta + y_\rho y_\theta)) = \frac{\partial}{\partial\theta}(\rho(x_\rho{}^2 + y_\rho{}^2))$$

and so we can define $A(\rho,\theta)$ as in (3.9.7) and (3.9.8) by

$$A_\rho = \rho(x_\rho{}^2 + y_\rho{}^2) \qquad (6.4.3)$$

$$A_\theta = \rho(x_\rho x_\theta + y_\rho y_\theta) . \qquad (6.4.4)$$

Then $\quad A_\rho x_\theta - A_\theta x_\rho = -\rho^3\cos\theta\left(\frac{y_\rho}{x}\right) \qquad (6.4.5)$

and $\quad A_\rho y_\theta - A_\theta y_\rho = \rho^3\cos\theta\left(\frac{x_\rho}{x}\right) . \qquad (6.4.6)$

Elimination of x_θ from (6.4.4) and (6.4.2) gives

$$y_\theta = \frac{A_\theta y_\rho}{A_\rho} + \frac{\rho^3\cos\theta\, x_\rho}{xA_\rho} . \qquad (6.4.7)$$

Similarly, $\quad x_\theta = \frac{A_\theta x_\rho}{A_\rho} - \frac{\rho^3\cos\theta\, y_\rho}{xA_\rho} . \qquad (6.4.8)$

If we differentiate (6.4.5) and (6.4.6) with respect to ρ and substitute for $x_{\rho\theta}$ and $y_{\rho\theta}$ in (6.4.1) then

$$\frac{x_\rho}{A_\rho}\{A_{\rho\theta}x_\rho + A_\theta x_{\rho\rho} - A_{\rho\rho}x_\theta - \frac{\partial}{\partial\rho}\left(\frac{\rho^3\cos\theta\, y_\rho}{x}\right)\} - x_\theta x_{\rho\rho} - \frac{x_\rho x_\theta}{\rho} \qquad (6.4.9)$$
$$+ \frac{y_\rho}{A_\rho}\{A_{\rho\theta}y_\rho + A_\theta y_{\rho\rho} - A_{\rho\rho}y_\theta + \frac{\partial}{\partial\rho}\left(\frac{\rho^3\cos\theta x_\rho}{x}\right)\} - y_\theta y_{\rho\rho} - \frac{y_\rho y_\theta}{\rho} = 0 .$$

Substitute for x_θ and y_θ from (6.4.7) and (6.4.8) in (6.4.9)

to/

to get

$$\frac{2\rho^3\cos\theta}{x}\,(x_{\rho\rho}y_\rho - x_\rho y_{\rho\rho}) = -A_\rho^{\,2}\,\frac{\partial}{\partial\rho}\left(\frac{A_\theta}{\rho A_\rho}\right)$$

$$\Rightarrow \frac{\partial}{\partial\rho}\left(\tan^{-1}\frac{y_\rho}{x_\rho}\right) = \frac{xA_\rho}{2\rho^2\cos\theta}\,\frac{\partial}{\partial\rho}\left(\frac{A_\theta}{\rho A_\rho}\right) .$$

Let $$B_\rho = \frac{xA_\rho}{\rho^2\cos\theta}\,\frac{\partial}{\partial\rho}\left(\frac{A_\theta}{\rho A_\rho}\right) . \qquad (6.4.10)$$

It might be supposed that this would be invalid at $\theta = -\pi/2$ but in this case either $x = 0$ or $\partial/\partial\rho(A_\theta/\rho A_\rho) = 0$ as well and some sort of limiting process is involved.

Then $y_\rho/x_\rho = \tan(B/2 + H(\theta))$ for some function $H(\theta)$.

But as $\rho \to \infty$, $x \to \rho\cos\theta$, $y \to \rho\sin\theta$, $A \to \rho^2/2$, $B \to A_\theta/\rho^2 = O(1/\rho^4)$ and so $H(\theta) \equiv \theta$. We now have

$$\frac{y_\rho}{x_\rho} = \tan(B/2 + \theta) . \qquad (6.4.11)$$

Since $x_\rho^{\,2} + y_\rho^{\,2} = A_\rho/\rho$ we can write

$$\left.\begin{aligned} x_\rho &= \frac{A_\rho}{\rho}\cos(B/2 + \theta) \\ y_\rho &= \frac{A_\rho}{\rho}\sin(B/2 + \theta) . \end{aligned}\right\} \qquad (6.4.12)$$

This looks identical to the corresponding result (3.9.23) for the two-dimensional case but now B is a different function since B_ρ is given by (6.4.10) instead of the previous definition which was $B_\rho = (A_\rho/\rho)\,\partial/\partial\rho\,(A_\theta/\rho A_\rho)$.

From (6.4.7) and (6.4.8)

$$\left.\begin{aligned} y_\theta &= \frac{A_\theta}{A_\rho}\sqrt{\frac{A_\rho}{\rho}}\,\sin(B/2+\theta) + \frac{\rho}{B_\rho}\frac{\partial}{\partial\rho}\left(\frac{A_\theta}{\rho A_\rho}\right)\sqrt{\frac{A_\rho}{\rho}}\,\cos(B/2+\theta) \\ x_\theta &= \frac{A_\theta}{A_\rho}\sqrt{\frac{A_\rho}{\rho}}\,\cos(B/2+\theta) - \frac{\rho}{B_\rho}\frac{\partial}{\partial\rho}\left(\frac{A_\theta}{\rho A_\rho}\right)\sqrt{\frac{A_\rho}{\rho}}\,\sin(B/2+\theta) . \end{aligned}\right\} \qquad (6.4.13)$$

The quantities $x(\rho,\theta)$ and $y(\rho,\theta)$ are clearly given in terms of two functions $A(\rho,\theta)$ and $B(\rho,\theta)$ but now there is no obvious/

obvious way of eliminating $B(\rho,\theta)$ in order to obtain a single equation for $A(\rho,\theta)$. The strategy employed for the wedge problem breaks down.

Since the equations for the pressure (6.1.1) and (6.1.2) are identical in the two-dimensional and three-dimensional problems it follows as in (3.9.26) that the pressure $\pi(\rho,\theta)$ is given by

$$\pi = A - \rho A_\rho/2 . \tag{6.4.14}$$

By analogy with the two-dimensional problem we can define angles ϕ and ψ such that

$$\frac{y_\rho}{x_\rho} = \tan(\theta+\phi) \tag{6.4.15}$$

$$\frac{y_\theta}{x_\theta} = \tan(\theta+\psi) , \tag{6.4.16}$$

and ϕ and ψ have the same interpretation as before, if the convected rays and circles are confined to the meridian plane. From (6.4.11) it follows that

$$B = 2\phi . \tag{6.4.17}$$

From (6.4.13) we have $\rho \frac{\partial}{\partial\rho}\left(\frac{A_\theta}{\rho A_\rho}\right) = 2\phi_\rho \frac{A_\theta}{A_\rho}\tan(\psi-\phi)$

$$\Rightarrow \quad \frac{A_\theta}{\rho A_\rho} = \exp\left\{2\int^\rho \phi_\rho \tan(\psi-\phi)d\rho\right\} . \tag{6.4.18}$$

This would give the angle ψ and hence $(\psi-\phi)$ if A and B were known.

It is possible to calculate what happens at large distances from the origin. If the behaviour at infinity is assumed to be like that due to a three-dimensional dipole then it can be shown (see Appendix II) that, for large ρ,

x/

$$x \sim \rho \cos\theta - \frac{3m \sin\theta \cos\theta}{\rho^3},$$

$$y \sim \rho \sin\theta + \frac{m(1-3\sin^2\theta)}{\rho^3}.$$

Then

$$A_\rho = \rho(x_\rho{}^2 + y_\rho{}^2)$$

$$\sim \rho + \frac{12m \sin\theta}{\rho^3},$$

$$A_\theta \sim - \frac{6m \cos\theta}{\rho^2}.$$

Hence $$A \sim \frac{\rho^2}{2} - \frac{6m \sin\theta}{\rho^2} \qquad (6.4.19)$$

if we are to satisfy $\pi = A - \rho A_\rho/2$ and $\pi \to 0$ as $\rho \to \infty$.

It follows immediately that

$$\pi \sim - \frac{12m \sin\theta}{\rho^2} \quad \text{as} \quad \rho \to \infty .$$

Now $xA_\rho/\rho^2\cos\theta \sim 1 + 9m \sin\theta/\rho^4$ and $A_\theta/\rho A_\rho \sim -6m \cos\theta/\rho^4$.

Hence, from (6.4.10) $B \sim A_\theta/\rho A_\rho$

$$\Rightarrow B \sim - \frac{6m \cos\theta}{\rho^4} . \qquad (6.4.20)$$

From (6.4.17) and (6.4.20)

$$\phi = \frac{B}{2} \sim - \frac{3m \cos\theta}{\rho^4} . \qquad (6.4.21)$$

Now

$$\frac{y_\theta}{x_\theta} \sim - \cot\theta + \frac{6m \cos\theta}{\rho^4}$$

$$\sim - \cot\left(\theta + \frac{6m \cos\theta \sin^2\theta}{\rho^4}\right)$$

$$= \tan\left(\frac{\pi}{2} + \theta + \frac{6m \cos\theta \sin\theta}{\rho^4}\right) .$$

From (6.4.16) we have

$$\psi \sim \frac{\pi}{2} + \frac{6m \cos\theta \sin^2\theta}{\rho^4} \qquad (6.4.22)$$

In this case there is no simple result like $\psi \sim \pi/2-\phi$ which we had for the wedge problem (cf. 3.4.8). Neither is there any explicit relation between ϕ and ψ on the free surface/

surface. However (6.4.22) shows that for large ρ, $(\psi-\pi/2)$ is unchanged both on the axis of symmetry, as for the wedge problem, and also on the free surface which is a new departure.

(6.5) Linearised theory.

Mackie (1962) has shown that the velocity potential for the linearised water entry of a cone is

$$\phi(x,y) = \frac{\varepsilon^2}{2}\{(1-y)\sinh^{-1}(\frac{1-y}{x})-(1+y)\sinh^{-1}(\frac{1+y}{x})+2\sinh^{-1}(\frac{y}{x})$$

$$+\sqrt{(1+y)^2+x^2} + \sqrt{(1-y)^2+x^2}\}$$

where ε is the semi-angle of the cone in the meridian plane.

Hence $\dfrac{\partial\phi}{\partial x} = \dfrac{\varepsilon^2}{2x}\{\sqrt{x^2+(1+y)^2} - \sqrt{x^2+(1-y)^2} - \dfrac{2y}{\sqrt{x^2+y^2}}\}$,

and $\dfrac{\partial\phi}{\partial y} = \dfrac{\varepsilon^2}{2}\{\dfrac{2}{\sqrt{x^2+y^2}} - \sinh^{-1}(\dfrac{1-y}{x}) - \sinh^{-1}(\dfrac{1+y}{x})\}$.

In dimensional coordinates these are equivalent to

$$\frac{\partial\Phi}{\partial X} = \frac{\varepsilon^2 V_o}{2X}\{\sqrt{X^2+(V_o t+Y)^2} - \sqrt{X^2+(V_o t-Y)^2} - \frac{2YV_o t}{\sqrt{X^2+Y^2}}\}.$$

$$\frac{\partial\Phi}{\partial Y} = \frac{\varepsilon^2 V_o}{2}\{\frac{2V_o t}{\sqrt{X^2+Y^2}} - \sinh^{-1}(\frac{V_o t-Y}{X}) - \sinh^{-1}(\frac{V_o t+Y}{X})\}.$$

To find the Lagrangian representation of this flow it is sufficient to solve

$$\frac{\partial X}{\partial t} = \frac{\partial\Phi}{\partial X} \text{ and } \frac{\partial Y}{\partial t} = \frac{\partial\Phi}{\partial Y}$$

subject to $X = A$ and $Y = B$ at $t = 0$. These give $X = A + O(\varepsilon^2)$ and $Y = B + O(\varepsilon^2)$ so that, to the required order of approximation we can write

$\partial X/$

$$\frac{\partial X}{\partial t}(A,B,t) = \frac{\varepsilon^2 V_o}{2A}\left\{ \sqrt{A^2+(V_o t+B)^2} - \sqrt{A^2+(V_o t-B)^2} - \frac{2BV_o t}{\sqrt{A^2+B^2}} \right\}.$$

$$\frac{\partial Y}{\partial t}(A,B,t) = \frac{\varepsilon^2 V_o}{2}\left\{\frac{2V_o t}{\sqrt{A^2+B^2}} - \sinh^{-1}\left(\frac{V_o t-B}{A}\right) - \sinh^{-1}\left(\frac{V_o t+B}{A}\right)\right\}.$$

Integration of these last two equations gives

$$X=A+\frac{\varepsilon^2}{4}\left\{\frac{V_o t+B}{A}\sqrt{A^2+(V_o t+B)^2} - \frac{V_o t-B}{A}\sqrt{A^2+(V_o t-B)^2} - \frac{2BV_o^2 t^2}{A\sqrt{A^2+B^2}} - \frac{2B}{A}\sqrt{A^2+B^2}\right.$$

$$\left. + A\sinh^{-1}\left(\frac{V_o t+B}{A}\right) - A\sinh^{-1}\left(\frac{V_o t-B}{A}\right) - 2A\sinh^{-1}\frac{B}{A}\right\}.$$

$$Y=B+\frac{\varepsilon^2}{2}\left\{2B\sinh^{-1}\left(\frac{B}{A}\right) - (V_o t+B)\sinh^{-1}\left(\frac{V_o t+B}{A}\right) - (V_o t-B)\sinh^{-1}\left(\frac{V_o t-B}{A}\right)\right.$$

$$\left. + \sqrt{A^2+(V_o t+B)^2} + \sqrt{A^2+(V_o t-B)^2} - 2\sqrt{A^2+B^2} + \frac{V_o^2 t^2}{\sqrt{A^2+B^2}}\right\}.$$

In dimensionless coordinates these can be replaced by

$$x = a + \frac{\varepsilon^2}{4}\left\{\frac{1+b}{a}\sqrt{a^2+(1+b)^2} - \frac{1-b}{a}\sqrt{a^2+(1-b)^2} - \frac{2b}{a\sqrt{a^2+b^2}} - \frac{2b}{a}\sqrt{a^2+b^2}\right.$$

$$\left. + a\sinh^{-1}\left(\frac{1+b}{a}\right) - a\sinh^{-1}\left(\frac{1-b}{a}\right) - 2a\sinh^{-1}\left(\frac{b}{a}\right)\right\} \qquad (6.5.1)$$

$$y = b + \frac{\varepsilon^2}{2}\left\{2b\sinh^{-1}\left(\frac{b}{a}\right) - (1+b)\sinh^{-1}\left(\frac{1+b}{a}\right) - (1-b)\sinh^{-1}\left(\frac{1-b}{a}\right)\right.$$

$$\left. + \sqrt{a^2+(1+b)^2} + \sqrt{a^2+(1-b)^2} - 2\sqrt{a^2+b^2} + \frac{1}{\sqrt{a^2+b^2}}\right\}. \qquad (6.5.2)$$

We can see that these forms for x and y satisfy some, but not all, of the required boundary conditions. On $b = 0$, $x = a$, $x_a = 1$, $y = \varepsilon^2(\sqrt{a^2+1} - \sinh^{-1}(1/a) - a + 1/2a)$ and so $x_a^2 + y_a^2 = 1$ as required by (6.2.1), and the equation of the free surface is

$$y = \varepsilon^2(\sqrt{x^2+1} - \sinh^{-1}(1/x) - x + 1/2x),$$

which agrees with the form given by Mackie (1962). As $a \to 0$,

b/

$b < -1$, $x \to 0$ as required by (6.2.3) with $\ell = 1$. It is easy to check that the asymptotic behaviour of x and y agrees with (6.2.4) where $m = \varepsilon^2/6$. However, as $a \to 0$, if $-1 < b < 0$, then both x and y become infinite and (6.2.2) cannot be satisfied. To elucidate this matter, consider linearised versions of equations (6.1.3) and (6.1.6). Let $x = a + \varepsilon^n \bar{x}$ and $y = b + \varepsilon^n \bar{y}$ for some positive integer n. Then, if we retain in the equations of motion only terms in ε^n,

$$a\bar{x}_{ab} + b\bar{x}_{bb} = a\bar{y}_{aa} + b\bar{y}_{ab} ,$$
$$a\bar{x}_a + a\bar{y}_b + \bar{x} = 0 .$$

Elimination of $\bar{y}$ leads to an equation for $\bar{x}$:

$$a\bar{x}_{aaa} + b\bar{x}_{aab} + a\bar{x}_{abb} + b\bar{x}_{bbb} + 2\bar{x}_{aa} + \frac{b}{a}\bar{x}_{ab} + \bar{x}_{bb} - \frac{\bar{x}_a}{a} - \frac{b\bar{x}_b}{a^2} + \frac{\bar{x}}{a^2} = 0 .$$

This equation may be written as

$$\rho\lambda_\rho + \lambda = 0 ,$$

where $\lambda = \bar{x}_{aa} + \bar{x}_{bb} + \bar{x}_a/a - \bar{x}/a^2$, $a = \rho\cos\theta$ and $b = \rho\sin\theta$.

Hence $\lambda = f(\theta)/_\rho$ for some function $f(\theta)$, and if we are to satisfy (6.2.4) the only possibility is

$f(\theta) \equiv 0$, and $\bar{x}$ must satisfy

$$\bar{x}_{aa} + \bar{x}_{bb} + \frac{\bar{x}_a}{a} - \frac{\bar{x}}{a^2} = 0 .$$

The boundary conditions may be deduced from (6.2.1)-(6.2.4) and are

$$\bar{x}(a,0) = 0 ,$$
$$\bar{x}(0,b) = 0 \quad \text{for } b < -1 ,$$
$$\bar{x} = O(\rho^{-3}) \text{ and } \bar{y} = O(\rho^{-3}) \text{ as } \rho \to \infty ,$$
$$\varepsilon^n \bar{x} = \varepsilon(b+1) \quad \text{for } a = 0,\ -1 < b < 0 .$$

The/

The obvious choice for n would seem to be $n = 1$ so that $\overline{x}(0,b) = b+1$ for $a = 0$, $-1 < b < 0$. However, if the differential equation for $\overline{x}$ has a bounded solution as $a \to 0$ then the presence of the term $\overline{x}/a^2$ shows that $\overline{x} \to 0$ as $a \to 0$ and we cannot satisfy $\overline{x}(0,b) = b+1$. This is analogous to the situation which arises in standard slender body theory where linearisation leads to the impossible task of solving

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = 0$$

with $\partial\phi/\partial r \neq 0$ on some portion of $r = 0$.

In the present cone water-entry problem the function $\overline{x}(a,b)$ given by (6.5.1) where n is now equal to 2 satisfies the required differential equation, and all but one of the boundary conditions, and breaks down in a neighbourhood of the line segment $a = 0$, $-1 \leqslant b \leqslant 0$.

In polar coordinates (6.5.1) and (6.5.2) become

$$x = \rho\cos\theta + \frac{\varepsilon^2}{4}\Big\{\frac{1+\rho\sin\theta}{\rho\cos\theta}\sqrt{1+2\rho\sin\theta+\rho^2} - \frac{1-\rho\sin\theta}{\rho\cos\theta}\sqrt{1-2\rho\sin\theta+\rho^2} - \frac{2\tan\theta}{\rho}$$
$$- 2\rho\tan\theta + \rho\cos\theta\Big(\sinh^{-1}\Big(\frac{1+\rho\sin\theta}{\rho\cos\theta}\Big) - \sinh^{-1}\Big(\frac{1-\rho\sin\theta}{\rho\cos\theta}\Big) - 2\sinh^{-1}(\tan\theta)\Big)\Big\} \qquad (6.5.3)$$

$$y = \rho\sin\theta + \frac{\varepsilon^2}{2}\Big\{2\rho\sin\theta\sinh^{-1}(\tan\theta) - (1+\rho\sin\theta)\sinh^{-1}\Big(\frac{1+\rho\sin\theta}{\rho\cos\theta}\Big)$$
$$-(1-\rho\sin\theta)\sinh^{-1}\Big(\frac{1-\rho\sin\theta}{\rho\cos\theta}\Big) + \sqrt{1+2\rho\sin\theta+\rho^2} + \sqrt{1-2\rho\sin\theta+\rho^2}$$
$$- 2\rho + \frac{1}{\rho}\Big\}. \qquad (6.5.4)$$

Then/

Then $x_\rho = \cos\theta + \frac{\varepsilon^2}{4}\{\cos\theta\left(\sinh^{-1}(\frac{1+\rho\sin\theta}{\rho\cos\theta}) - \sinh^{-1}(\frac{1-\rho\sin\theta}{\rho\cos\theta}) - 2\sinh^{-1}(\tan\theta)\right)$

$- 2\tan\theta + \frac{2\tan\theta}{\rho^2} + \frac{\rho\sin\theta - 1}{\rho^2\cos\theta}\sqrt{1+2\rho\sin\theta+\rho^2} + \frac{\rho\sin\theta+1}{\rho^2\cos\theta}\sqrt{1-2\rho\sin\theta+\rho^2}\}$

$y_\rho = \sin\theta + \frac{\varepsilon^2}{2}\{-\sin\theta\{\sinh^{-1}(\frac{1+\rho\sin\theta}{\rho\cos\theta}) - \sinh^{-1}(\frac{1-\rho\sin\theta}{\rho\cos\theta}) - 2\sinh^{-1}(\tan\theta)\}$

$- 2 - \frac{1}{\rho^2} + \frac{1}{\rho}\sqrt{1+2\rho\sin\theta+\rho^2} + \frac{1}{\rho}\sqrt{1-2\rho\sin\theta+\rho^2}\}$

and $x_{\rho\rho} = \frac{\varepsilon^2}{2}\{\frac{\rho\sin\theta+1}{\rho^3\cos\theta\sqrt{1+2\rho\sin\theta+\rho^2}} + \frac{\rho\sin\theta - 1}{\rho^3\cos\theta\sqrt{1-2\rho\sin\theta+\rho^2}} - \frac{2\tan\theta}{\rho^3}\}$

$$y_{\rho\rho} = \frac{\varepsilon^2}{2}\{\frac{-1}{\rho^2\sqrt{1+2\rho\sin\theta+\rho^2}} - \frac{1}{\rho^2\sqrt{1-2\rho\sin\theta+\rho^2}} + \frac{2}{\rho^3}\} .$$

Now from (3.9.24) and (3.9.25)

$$\pi_\rho = -\rho^2(x_\rho x_{\rho\rho} + y_\rho y_{\rho\rho}) \quad \text{and} \quad \pi_\theta = -\rho^2(x_{\rho\rho}x_\theta + y_{\rho\rho}y_\theta) .$$

Hence $\pi_\rho = \frac{\varepsilon^2}{2}\{-\frac{1}{\rho\sqrt{1+2\rho\sin\theta+\rho^2}} + \frac{1}{\rho\sqrt{1-2\rho\sin\theta+\rho^2}}\}$

and $\pi_\theta = \frac{\varepsilon^2}{2}\{\frac{\rho+\sin\theta}{\cos\theta\sqrt{1+2\rho\sin\theta+\rho^2}} + \frac{\rho-\sin\theta}{\cos\theta\sqrt{1-2\rho\sin\theta+\rho^2}} - 2\sec\theta\}$.

Since $\pi \to 0$ as $\rho \to \infty$,

$$\pi = \frac{\varepsilon^2}{2}\{\sinh^{-1}(\frac{1+\rho\sin\theta}{\rho\cos\theta}) - \sinh^{-1}(\frac{1-\rho\sin\theta}{\rho\cos\theta}) - 2\sinh^{-1}(\tan\theta)\}. \quad (6.5.5)$$

From (6.4.14) $\pi = A - \rho A_\rho/2$.

Hence $\frac{A}{\rho^2} = \int^\rho -\frac{2\pi}{\rho^3}\, d\rho$.

$\Rightarrow \frac{A}{\rho^2} = \frac{\varepsilon^2}{4}\{(3\sin^2\theta - 1 - \frac{2}{\rho^2})\left(\sinh^{-1}(\frac{1-\rho\sin\theta}{\rho\cos\theta}) - \sinh^{-1}(\frac{1+\rho\sin\theta}{\rho\cos\theta})\right)$

$- \frac{4\sinh^{-1}(\tan\theta)}{\rho^2} + \sqrt{1+2\rho\sin\theta+\rho^2}\,(\frac{3\sin\theta}{\rho} - \frac{1}{\rho^2})$

$+ \sqrt{1-2\rho\sin\theta+\rho^2}\,(\frac{3\sin\theta}{\rho} + \frac{1}{\rho^2}) + f(\theta)$

for some function $f(\theta)$.

We know from (6.4.19) that for large ρ ,

A/

$$A \sim \frac{\rho^2}{2} - \frac{6m \sin\theta}{\rho^2}\ , \quad \text{where in the linearised theory}$$

$$m = \varepsilon^2/6.$$

It follows that the correct form for $A(\rho,\theta)$ is

$$A = \frac{\rho^2}{2} + \frac{\varepsilon^2}{4}\Big\{\big(\rho^2(3\sin^2\theta-1)-2\big)\Big(\sinh^{-1}\big(\frac{1-\rho\sin\theta}{\rho\cos\theta}\big)-\sinh^{-1}\big(\frac{1+\rho\sin\theta}{\rho\cos\theta}\big)+2\sinh^{-1}(\tan\theta)\Big)$$

$$- 6\rho^2\sin\theta + (3\rho\sin\theta-1)\sqrt{1+2\rho\sin\theta+\rho^2}+(3\rho\sin\theta+1)\sqrt{1-2\rho\sin\theta+\rho^2}\Big\}. \qquad (6.5.6)$$

We can find $B(\rho,\theta)$ from (6.4.10).

Since $x = \rho\cos\theta + O(\varepsilon^2)$ and $A_\rho = \rho + O(\varepsilon^2)$ it follows that $B = A_\theta/\rho^2$, to the required degree of accuracy.

From (6.5.6) we find that

$$A_\theta = \frac{\varepsilon^2}{4}\Big\{6\rho^2\sin\theta\cos\theta\Big(\sinh^{-1}\big(\frac{1-\rho\sin\theta}{\rho\cos\theta}\big)-\sinh^{-1}\big(\frac{1+\rho\sin\theta}{\rho\cos\theta}\big)+ 2\sinh^{-1}(\tan\theta)\Big)$$

$$+ \frac{2\big(\rho(2-3\sin^2\theta)+\sin\theta\big)}{\cos\theta}\sqrt{1+2\rho\sin\theta+\rho^2} + \frac{2\big(\rho(2-3\sin^2\theta)-\sin\theta\big)}{\cos\theta}\sqrt{1-2\rho\sin\theta+\rho^2}$$

$$+ \frac{\rho^2(12\sin^2\theta-8)}{\cos\theta} - \frac{4}{\cos\theta}\Big\}.$$

Hence, $B = \frac{\varepsilon^2}{2}\Big\{3\sin\theta\cos\theta\Big(\sinh^{-1}\big(\frac{1-\rho\sin\theta}{\rho\cos\theta}\big)-\sinh^{-1}\big(\frac{1+\rho\sin\theta}{\rho\cos\theta}\big)+2\sinh^{-1}(\tan\theta)\Big)$

$$+ \frac{\big(\rho(2-3\sin^2\theta)+\sin\theta\big)}{\rho^2\cos\theta}\sqrt{1+2\rho\sin\theta+\rho^2}+ \frac{\big(\rho(2-3\sin^2\theta)-\sin\theta\big)}{\rho^2\cos\theta}\sqrt{1-2\rho\sin\theta+\rho^2}$$

$$+ \frac{6\sin^2\theta-4}{\cos\theta} - \frac{2}{\rho^2\cos\theta}\Big\}. \qquad (6.5.7)$$

The angle ϕ can be found at once from (6.4.17) and $\phi = B/2$.

From (6.4.16), $\tan(\theta+\psi) = y_\theta/x_\theta$ and so

$$\psi = \pi/2 - \frac{\varepsilon^2}{2}\Big\{\sin\theta\cos\theta\Big(\sinh^{-1}\big(\frac{1-\rho\sin\theta}{\rho\cos\theta}\big)-\sinh^{-1}\big(\frac{1+\rho\sin\theta}{\rho\cos\theta}\big)+2\sinh^{-1}(\tan\theta)\Big)$$

$$- \frac{\sin^2\theta}{\rho\cos\theta}\sqrt{1+2\rho\sin\theta+\rho^2} - \frac{\sin^2\theta}{\rho\cos\theta}\sqrt{1-2\rho\sin\theta+\rho^2} + \frac{2\sin^2\theta}{\cos\theta}\Big\}. \qquad (6.5.8)$$

It is easy to check that these functions B, ϕ and ψ have asymptotic behaviour consistent with (6.4.20)-(6.4.22) where

m/

$m = \epsilon^2/6$. There is no simple relation between ϕ and ψ either on the free surface, or for large values of ρ , in contrast with the wedge problem.

APPENDIX I

Behaviour near the tip of the wedge.

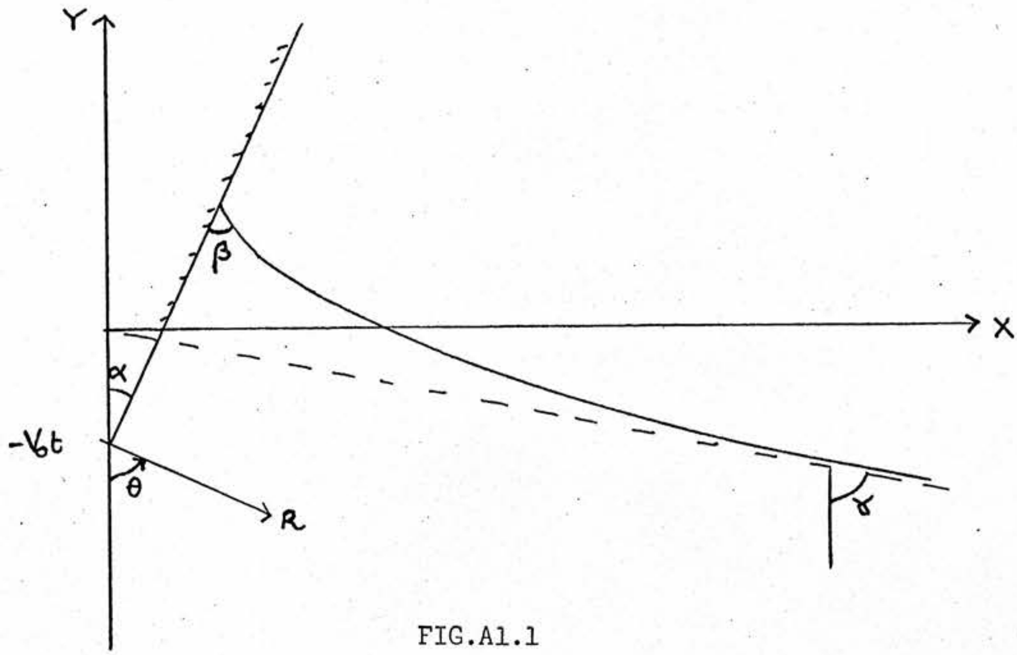


FIG.A1.1

Consider the water entry problem as shown in Figure A1.1. Clearly the asymptotic form of the free surface does not affect the local flow near the tip of the wedge and so there is no need to consider only the problem of a wedge entering a half-space of liquid. Let $X = R \sin \theta$ and $Y = -R \cos \theta - V_0 t$. If $\Phi(X, Y, t)$ is the velocity potential, then, near the tip of the wedge,

$$\Phi(X, Y, t) = -V_0 Y + C(t)\, R^{\omega} \cos \omega\theta + o(R^{\omega})$$

where $\omega = \pi/(\pi-\alpha)$. Then Φ satisfies $\partial\Phi/\partial\theta = 0$ on $\theta = 0$ and on $\theta = \pi-\alpha$, and $\partial\Phi/\partial X = 0$, $\partial\Phi/\partial Y = -V_0$ at $Y = -V_0 t$. We note that, if $0 < \alpha < \pi/2$, then $1 < \omega < 2$. By similarity we can write $C(t) = C\, V_0^{2-\omega} t^{1-\omega}$

for/

for some dimensionless constant C.* For convenience we shall omit the term $o(R^{\omega})$ and investigate the consequences of a velocity potential of the form

$$\Phi(X, Y, t) = -V_o Y + C\, V_o^{2-\omega}\, t^{1-\omega}\, R^{\omega} \cos \omega\theta \qquad (A1.1)$$

and regard the result as an approximation to the local flow near the tip of the wedge.

Now from (A1.1)

$$\frac{\partial\Phi}{\partial X} = \omega\, C(t)\, R^{\omega-2} \, (X \cos \omega\theta + (Y + V_o t) \sin \omega\theta)$$

$$\frac{\partial\Phi}{\partial Y} = -V_o + \omega\, C(t)\, R^{\omega-2} \, ((Y + V_o t) \cos \omega\theta - X \sin \omega\theta).$$

To find the particle paths we have to solve

$$\frac{dX}{dt} = \frac{\partial\Phi}{\partial X} \qquad \text{and} \qquad \frac{dY}{dt} = \frac{\partial\Phi}{\partial Y}$$

subject to suitable boundary conditions. It is a simple matter to do this for particles initially on the axis of symmetry since we know that $X(0, B, t_B) = 0$ and $Y(0, B, t_B) = -V_o t_B$ (see section 3.4) and these constitute the necessary boundary conditions.

For $t < t_B$, we have

$$\frac{dX}{dt}(0, B, t) = 0$$

$$\frac{dY}{dt}(0, B, t) = -V_o - \omega\, C(t)(-(Y + V_o t))^{\omega-1} .$$

These may be integrated to give

$$X(0, B, t) = 0$$

$$Y(0, B, t) = -V_o t - (-\omega C)^{1/(2-\omega)}\, V_o(-t^{2-\omega} + t_B^{2-\omega})^{1/(2-\omega)} .$$

In the similarity variables, we have for $b < -\ell$,

$$x(0, b) = 0$$

$$y(0, b) = -1 - (-\omega C)^{1/(2-\omega)}(-1 + (-b/\ell)^{2-\omega})^{1/(2-\omega)} . \qquad (A1.2)$$

For/

*C must be negative if the flow relative to the tip is in the positive Y-direction.

For $t > t_B$ and $A = 0$, $X = R \sin\alpha$ and
$Y = R\cos\alpha - V_o t$,
where $\frac{dR}{dt} = -\omega C(t)\, R^{\omega-1}$,

$$\Rightarrow \quad R = (-\omega C)^{1/(2-\omega)}\, V_o (t^{2-\omega} - t_B^{2-\omega})^{1/(2-\omega)} .$$

Hence, for $b > -\ell$,

$$x(0, b) = (-\omega C)^{1/(2-\omega)} \sin\alpha\, (1 - (-b/\ell)^{2-\omega})^{1/(2-\omega)}$$

$$y(0, b) = -1 + (-\omega C)^{1/(2-\omega)} \cos\alpha (1 - (-b/\ell)^{2-\omega})^{1/(2-\omega)} . \quad \text{(A1.3)}$$

Differentiation of (A1.2) and (A1.3) shows that q and s are continuous and equal to zero at $b = -\ell$.

For $\alpha = \pi/2$, the above analysis does not hold. In this case $\omega = 2$, and for $t < t_B$ we have to solve

$$\frac{dR}{dt} = \frac{2CR}{t} ,$$

subject to $R = 0$ at $t = t_B$. Integration gives $R = \lambda t^{2C}$ for some constant λ, and there is no finite value of t for which $R = 0$. The particles on the axis of symmetry never actually reach the corner. This is the case in the impact problem, given by $\alpha = \pi/2$, $\gamma < \pi/2$, where particles on the axis of symmetry cannot move round the corner and one is led to assume that particles on the free surface are continually deposited on the wall. In this case one must ask whether the solution given by Cumberbatch (1960) is correct as he used the conservation of arc-length property on the whole of the free surface, and the above analysis shows that it should only be applied to a part of the initially undisturbed free surface. This is of course vitally important/

important when boundary conditions are to be formulated for a Lagrangian description of the impact problem, when it is essential to know what happens to boundary particles as the motion proceeds.

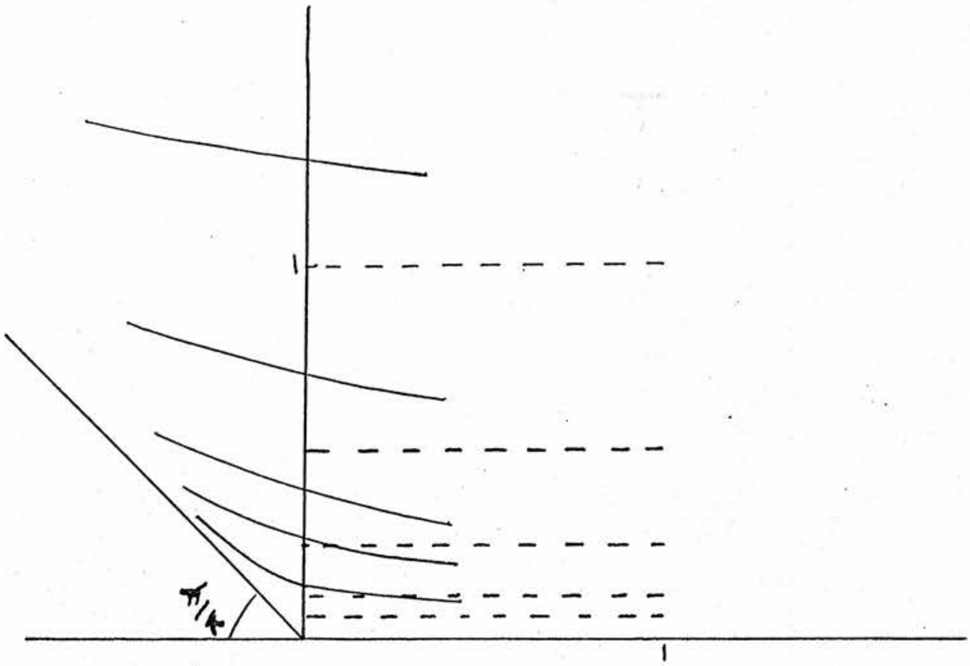
For $\alpha < \pi/2$, we now have a satisfactory solution for the local flow near the tip of the wedge. It is not possible to give an analytic solution for particles off the axis of symmetry since we do not know any suitable boundary conditions. A series of calculations was carried out, without using similarity variables, to see if the situation could be clarified. The equations to be solved are

$$\frac{d\xi}{dt} = \omega C(t)\, R^{\omega-2}(\xi \cos \omega\theta + \eta \sin \omega\theta)$$

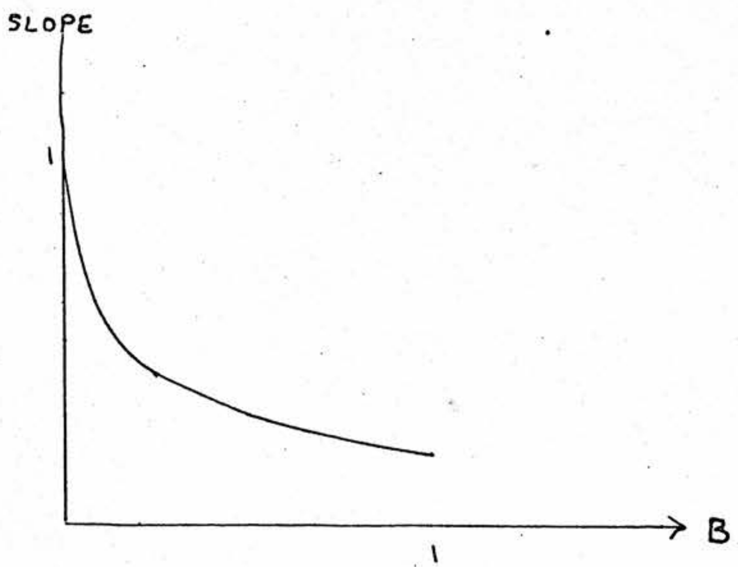
$$\frac{d\eta}{dt} = \omega C(t)\, R^{\omega-2}(\eta \cos \omega\theta - \xi \sin \omega\theta)$$

subject to $\xi = A$ and $\eta = B$ at $t = 0$, where $\xi = R \cos \theta$, $\eta = R \sin \theta$ and ω and $C(t)$ are to be chosen. The equations were solved by a Runge-Kutta method for values of $\alpha = \pi/36$, $\pi/4$ and $5\pi/12$, corresponding to $\omega = 36/35$, $4/3$ and $12/7$. The function $C(t)$ was chosen to be $-(1+t)^n$ for integral values of n between -2 and +2. A and B were taken to lie in the range $(0, 1)$, with more points near the origin than further away from it. As was mentioned in section 3.8, one of the objects of this calculation is to find out/

out what sort of discontinuity occurs in the slope of the curves B = constant at $A = 0$ as B approaches zero. Some typical results are illustrated in Figures (A1.2)-(A1.4). In no case was a noticeable discontinuity of slope found. A more detailed analysis where the number of points near $A = B = 0$ was increased failed to produce any more definite result.



DOTTED LINES DENOTE POSITION AT T = 0
SOLID LINES DENOTE POSITION AT T = 1



SLOPE OF LINES OF CONSTANT B AT A = 0

FIG. A1.2

$\alpha = \pi/4, \quad \omega = 4/3, \quad C(t) = -(1+t)^{-2}$

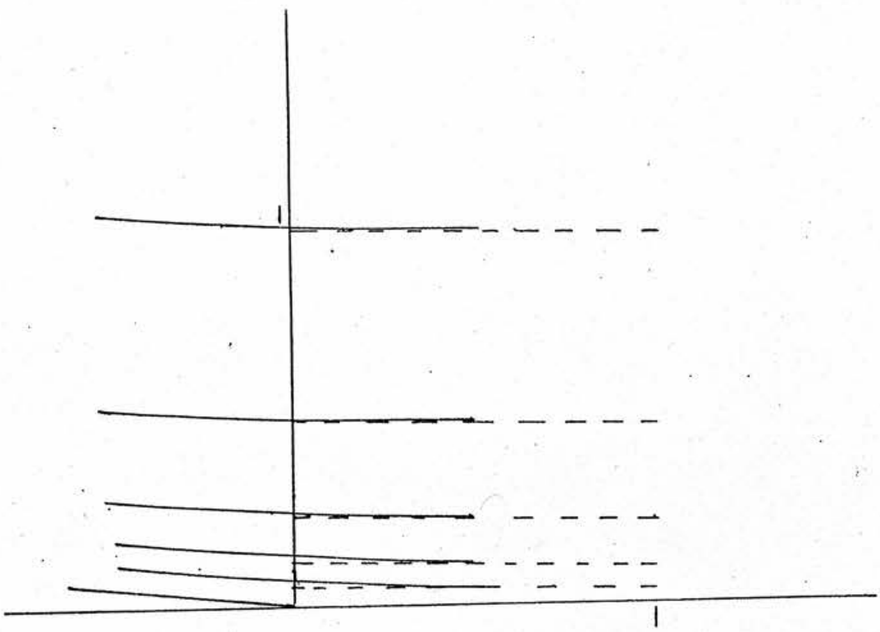


FIG. A1.3

$\alpha = \pi/36, \quad \omega = 36/35, \quad C(t) = -(1+t)^{-2}$

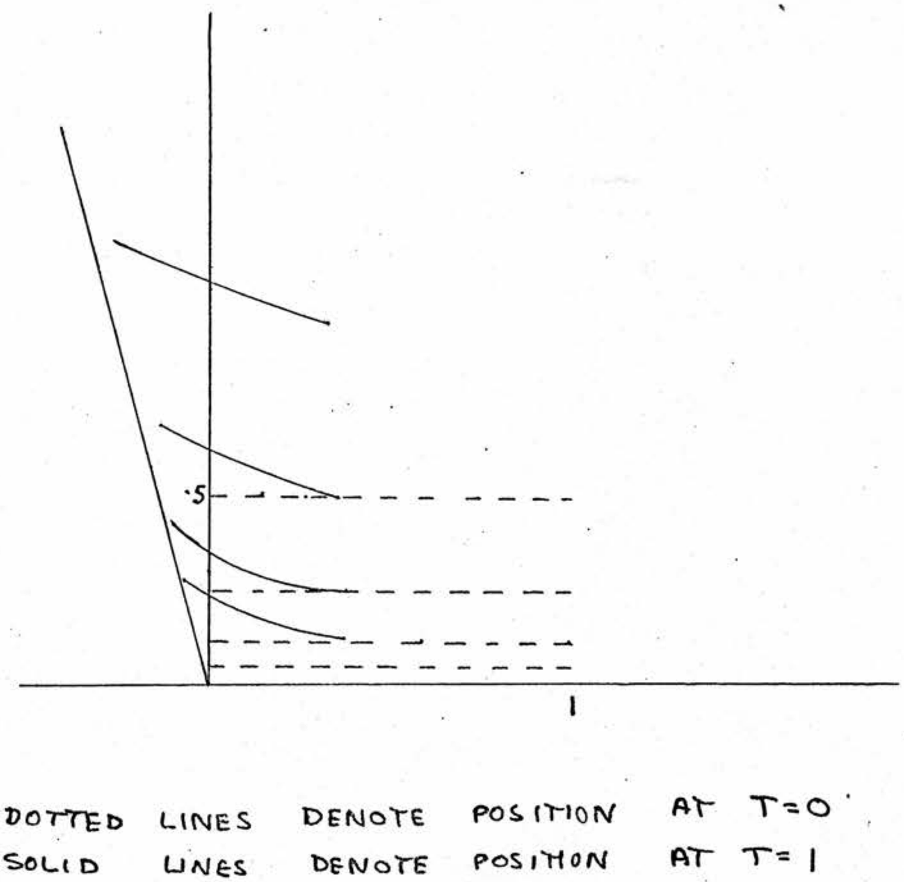
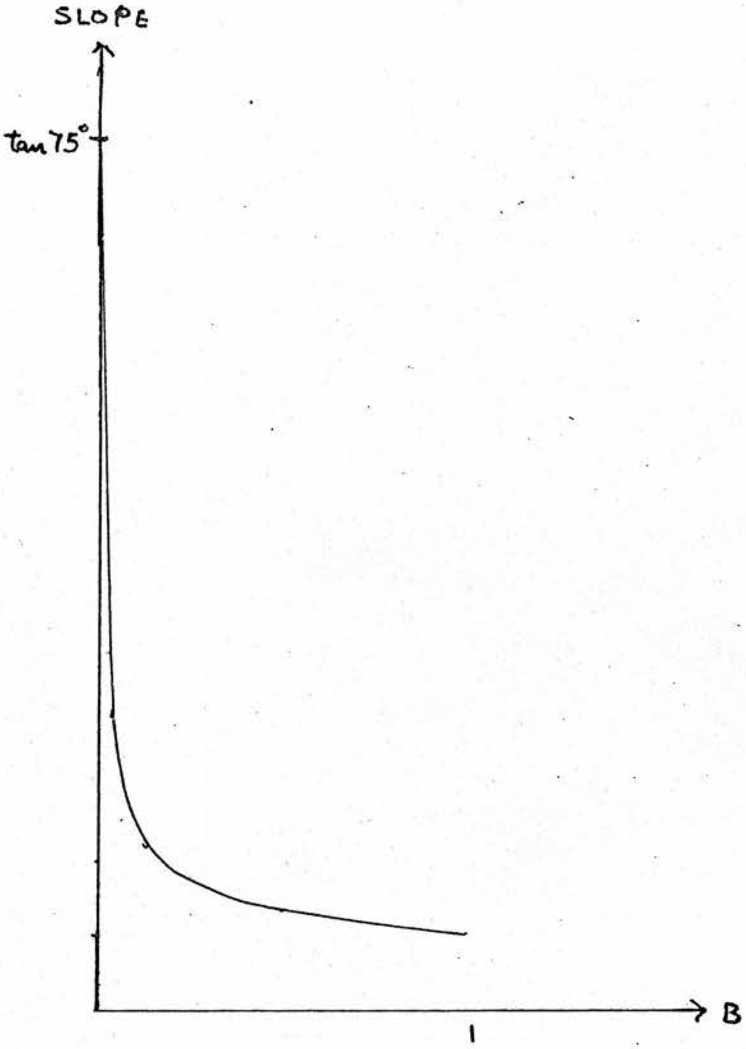


FIG. A1.4(a)

$\alpha = 5\pi/12, \quad \omega = 12/7, \quad C(t) = -(1+t)^{-2}$



SLOPE OF LINES OF CONSTANT B AT A = 0

FIG. A1.4(b)

$\alpha = 5\pi/12, \quad \omega = 12/7, \quad C(t) = -(1+t)^{-2}$

APPENDIX II

Lagrangian flow for a dipole

First of all consider a two-dimensional dipole, situated at the origin and pointing in the negative Y-direction. The velocity potential for this problem is

$$\Phi\ (R,\ \theta,\ t)\ =\ \frac{M(t)\ \sin\ \theta}{R} \qquad \text{(A2.1)}$$

where $X = R\cos\theta$, $Y = R\sin\theta$ and $M(t)$ is the strength of the dipole. If we are to be able to use the similarity variables then $M(t)$ must be written as $M(t) = m\ V_o^3\ t^2$ for some dimensionless constant m.

$$\text{Then}\quad \frac{\partial\Phi}{\partial R} = -\ \frac{m\ V_o^3\ t^2\ \sin\theta}{R^2}$$

$$\text{and}\quad \frac{1}{R}\frac{\partial\Phi}{\partial\theta} = \frac{m\ V_o^3\ t^2\ \cos\theta}{R^2}\ . \qquad \text{(A2.2)}$$

To find the Lagrangian flow we must solve

$$\frac{dR}{dt} = -\ \frac{m\ V_o^3\ t^2\ \sin\theta}{R^2}$$

$$R\,\frac{d\theta}{dt} = \frac{m\ V_o^3\ t^2\ \cos\theta}{R^2} \qquad \text{(A2.3)}$$

subject to $R = (A^2 + B^2)^{\frac{1}{2}}$ and $\theta = \tan^{-1}(B/A)$ at $t = 0$.

Division of the differential equations leads to

$$\frac{dR}{Rd\theta} = -\ \frac{\sin\theta}{\cos\theta}$$

$$\Rightarrow\ R = \frac{A^2 + B^2}{A}\cos\theta\ . \qquad \text{(A2.4)}$$

But/

But $R = V_o t r$ and so $r = \{(a^2+b^2)/a\} \cos\theta$.

Let $(a^2+b^2)^{\frac{1}{2}} = \rho$ and $\tan^{-1}(b/a) = \gamma$ so that ρ and γ are polar co-ordinates in the a-b plane.

Then $$r = \frac{\rho\cos\theta}{\cos\gamma} . \qquad (A2.5)$$

Substitute for R from (A2.4) in (A2.3) to get

$$\frac{d\theta}{dt} = \frac{m\, V_o^3\, A^3\, t^2}{(A^2+B^2)^3 \cos^2\theta}$$

$$\Rightarrow \quad \frac{\theta}{2} + \tfrac{1}{4}\sin 2\theta = \frac{m\cos^3\gamma}{3\rho^3} + \frac{\gamma}{2} + \tfrac{1}{4}\sin 2\gamma. \quad (A2.6)$$

The formulae (A2.5) and (A2.6) give an exact representation for r and θ as functions of ρ and γ for flow due to a dipole at the origin of strength $M = m\, V_o^3\, t^2$. What we are really interested in is an asymptotic representation for large values of ρ.

Write $\rho = {}^1/\varepsilon$ in (A2.6).

$$\frac{\theta}{2} + \tfrac{1}{4}\sin 2\theta = \frac{m\cos^3\gamma}{3}\,\varepsilon^3 + \frac{\gamma}{2} + \tfrac{1}{4}\sin 2\gamma .$$

We want an expression of the form $\theta(\varepsilon) = \theta(0)+\varepsilon\theta'(0)+ \ldots$.

At $\varepsilon = 0$, $\theta = \gamma$.

$$\cos^2\theta\,\frac{d\theta}{d\varepsilon} = (m\cos^3\gamma)\,\varepsilon^2.$$

At $\varepsilon = 0$, $\frac{d\theta}{d\varepsilon} = 0$.

$$-\sin 2\theta\left(\frac{d\theta}{d\varepsilon}\right)^2 + \cos^2\theta\,\frac{d^2\theta}{d\varepsilon^2} = (2m\cos^3\gamma)\varepsilon.$$

At $\varepsilon = 0$, $\frac{d^2\theta}{d\varepsilon^2} = 0$.

$$-2\cos 2\theta\left(\frac{d\theta}{d\varepsilon}\right)^3 - 3\sin 2\theta\left(\frac{d\theta}{d\varepsilon}\right)\left(\frac{d^2\theta}{d\varepsilon^2}\right) + \cos^2\theta\,\frac{d^3\theta}{d\varepsilon^3} = 2m\cos^3\gamma .$$

At $\varepsilon = 0$, $\frac{d^3\theta}{d\varepsilon^3} = 2m\cos\gamma$.

Hence/

Hence $\theta(\varepsilon) = \gamma + \frac{\varepsilon^3}{3}(m \cos \gamma) + \ldots.$

$$\Rightarrow \theta(\rho) \sim \gamma + \frac{m \cos \gamma}{3\rho^3} \quad \text{for large values of } \rho. \qquad (A2.7)$$

Then $\sin \theta \sim \sin \gamma + \dfrac{m \cos^2\gamma}{3\rho^3}$ and

$$\cos \theta \sim \cos \gamma - \frac{m \sin \gamma \cos \gamma}{3\rho^3} \quad .$$

From (A2.5) $r = \rho \cos \theta/\cos \gamma$ and so

$$x = r \cos \theta = \rho \cos^2\theta/\cos \gamma$$

$$\text{and} \quad y = r \sin \theta = \rho \sin \theta \cos \theta/\cos \gamma.$$

Thus $$x \sim \rho \cos \gamma - \frac{m \sin 2\gamma}{3\rho^2} \qquad (A2.8)$$

and $$y \sim \rho \sin \gamma + \frac{m \cos 2\gamma}{3\rho^2} \quad . \qquad (A2.9)$$

We know from (2.4.7) that in the linearised theory of wedge water entry

$$\phi(x, y) = \frac{\alpha}{2\pi} \int_0^1 \log \frac{x^2 + (y + \eta)^2}{x^2 + (y - \eta)^2} \, d\eta \quad \text{and so}$$

$$\phi(r, \theta) = \frac{\alpha}{2\pi} \int_0^1 \log\left(\frac{r^2 + 2 r \eta \sin \theta + \eta^2}{r^2 - 2 r \eta \sin \theta + \eta^2}\right) d\eta \, .$$

For large r, $\log\left(\dfrac{r^2 + 2 r \eta \sin \theta + \eta^2}{r^2 - 2 r \eta \sin \theta + \eta^2}\right) \sim \dfrac{4 \eta \sin \theta}{r} + O\left(\dfrac{1}{r^3}\right)$.

Hence $\phi(r, \theta) \sim \dfrac{\alpha}{\pi} \dfrac{\sin \theta}{r}$ for large r.

Comparing this with (A2.1) we see that in this case

$$m = \alpha/\pi \, . \qquad (A2.10)$$

The asymptotic behaviour of $x(\rho, \theta)$ and $y(\rho, \theta)$ for the wedge water-entry problem is then given by (A2.8) and (A2.9) and in the particular case of the linear theory the constant m is identified by (A2.10).

Next/

Next we may consider a three-dimensional dipole, again situated at the origin and pointing in the negative Y-direction where X and Y are now cylindrical polar co-ordinates. If the strength of the dipole is $M(t)$ then the velocity potential is

$$\Phi(R, \theta, t) = \frac{M(t) \sin \theta}{R^2} \qquad (A2.11)$$

where $X = R \cos \theta$ and $Y = R \sin \theta$. In this case we must have $M(t) = m V_o^4 t^3$.

$$\text{Then } \frac{\partial \Phi}{\partial R} = - \frac{2 m V_o^4 t^3 \sin \theta}{R^3}$$

$$\text{and } \frac{1}{R} \frac{\partial \Phi}{\partial \theta} = \frac{m V_o^4 t^3 \cos \theta}{R^3} . \qquad (A2.12)$$

We have to solve

$$\frac{dR}{dt} = - \frac{2 m V_o^4 t^3 \sin \theta}{R^3}$$

$$R\frac{d\theta}{dt} = \frac{m V_o^4 t^3 \cos \theta}{R^3} \qquad (A2.13)$$

subject to $R = (A^2 + B^2)^{\frac{1}{2}}$ and $\theta = \tan^{-1}(B/A)$ at $t = 0$.

Division of the differential equations gives

$$\frac{dR}{Rd\theta} = - \frac{2 \sin \theta}{\cos \theta}$$

$$\Rightarrow R = \frac{(A^2 + B^2)^{3/2}}{A^2} \cos^2\theta .$$

Since $R = V_o t r$, we have

$$r = \frac{\rho \cos^2\theta}{\cos^2\gamma} \qquad (A2.14)$$

where $a = \rho \cos \gamma$ and $b = \rho \sin \gamma$.

Substitute/

Substitute for R in (A2.13). Then

$$\frac{d\theta}{dt} = \frac{m V_o^4 A^8 t^3}{(A^2+B^2)^6 \cos^7\theta}$$

$$\Rightarrow \frac{1}{64}\left(35 \sin\theta + 7 \sin 3\theta + \frac{7}{5} \sin 5\theta + \frac{1}{7} \sin 7\theta\right) = \frac{m\cos^8\gamma}{4\rho^4}$$

$$+ \frac{1}{64}\left(35 \sin\gamma + 7 \sin 3\gamma + \frac{7}{5} \sin 5\gamma + \frac{1}{7} \sin 7\gamma\right). \quad \text{(A2.15)}$$

The formulae in (A2.14) and (A2.15) give an exact representation for the Lagrangian flow due to a dipole at the origin of strength $m V_o^4 t^3$. Again we really wish to consider the asymptotic flow for large ρ.

Write $\rho = {}^1/\varepsilon$ in (A2.15) and look for an expansion of the form $\theta(\varepsilon) = \theta(0) + \varepsilon\theta'(0) + \ldots.$

At $\varepsilon = 0$, $\theta = \gamma$, $\frac{d\theta}{d\varepsilon} = \frac{d^2\theta}{d\varepsilon^2} = \frac{d^3\theta}{d\varepsilon^3} = 0$ and

$$\frac{d^4\theta}{d\varepsilon^4} = 6 m \cos\gamma.$$

Hence $\theta \sim \gamma + \dfrac{m \cos\gamma}{4\rho^4}$ for large ρ. (A2.16)

From (A2.14) $r = \rho \cos^2\theta/\cos^2\gamma$ and so

$x = \rho \cos^3\theta/\cos^2\gamma$ and

$y = \rho \sin\theta \cos^2\theta/\cos^2\gamma$. From (A2.16)

$\sin\theta \sim \sin\gamma + \dfrac{m \cos^2\gamma}{4\rho^4}$ and $\cos\theta \sim \cos\gamma - \dfrac{m \sin\gamma\cos\gamma}{4\rho^4}$.

$$\text{Hence } x \sim \rho\cos\gamma - \frac{3 m \sin\gamma \cos\gamma}{4\rho^3} \quad \text{(A2.17)}$$

$$\text{and } y \sim \rho \sin\gamma + \frac{m(1 - 3\sin^2\gamma)}{4\rho^3}. \quad \text{(A2.18)}$$

Now we know from (2.4.9) that in the linearised theory for water entry of a cone

$$\phi(r, \theta) = \frac{\alpha^2}{2}\{(1-r \sin\theta) \sinh^{-1}\left(\frac{1-r \sin\theta}{r\cos\theta}\right) - (1+r\sin\theta)\sinh^{-1}\left(\frac{1+r\sin\theta}{r\cos\theta}\right) + 2\sinh^{-1}(\tan\theta) + \sqrt{r^2+2r\sin\theta+1} - \sqrt{r^2-2r\sin\theta+1}\}.$$

For/

For large r, $\phi(r, \theta) \sim \alpha^2 \sin\theta / 6r^2$.

Comparison with (A2.11) shows that

$$m = \alpha^2/6. \tag{A2.19}$$

The asymptotic behaviour of $x(\rho, \theta)$ and $y(\rho, \theta)$ for the cone water-entry problem is given by (A2.17) and (A2.18) and in the special case of the linear theory we can identify the constant m from (A2.19).

APPENDIX III

It has been shown in section (4.2) that in the water entry problem, the flow near the contact point is given by

$$\chi = -\tfrac{1}{2} r^2 + D_1 r^a \cos a(\tfrac{\pi}{2} + \alpha + \theta) + o(r^a)$$

and if the equation of the free surface is $\xi = h(\eta)$,

$$h(\eta) = - \eta \tan(\beta-\alpha) + c_m \eta^{a-1} + o(\eta^{a-1})$$

where $a = \pi/2\beta$ and $c_m = -D_1 a \sec^a(\beta-\alpha)/(a-2)$.
The object of this appendix is to calculate a second term in the expansions for χ and $h(\eta)$.

Let
$$\chi = -\tfrac{1}{2} r^2 + D_1 r^a \cos a(\tfrac{\pi}{2} + \alpha + \theta)$$
$$+ D_2 r^b \cos b(\tfrac{\pi}{2} + \alpha + \theta) + o(r^b) \qquad (A3.1)$$

thus satisfying condition (4.2.4) that

$(\partial\chi/\partial\xi) \cos \alpha = (\partial\chi/\partial\eta) \sin \alpha$ on $\theta = -(\pi/2 + \alpha)$.

Let
$$h(\eta) = - \eta \tan(\beta-\alpha) + c_m \eta^{a-1}$$
$$+ c_n \eta^n + o(\eta^n). \qquad (A3.2)$$

We must choose b, n, D_2 and c_n so that $b > a$, $n > a-1$ and the boundary conditions (4.2.6) and (4.2.7) are satisfied.

First of all we calculate χ and s^2 on the free surface.

On $\xi = h(\eta)$,

$$\xi^2 + \eta^2 = \eta^2 \sec^2(\beta-\alpha) - 2c_m \tan(\beta-\alpha) \eta^a + c_m^2 \eta^{2a-2}$$
$$- 2c_n \tan(\beta-\alpha) \eta^{n+1} + o(\eta^{n+1}) + O(\eta^{a+n-1}). \,(A3.3)$$

Hence/

Hence,

$$r^a = \eta^a \sec^a(\beta-\alpha) - a\, c_m \tan(\beta-\alpha) \sec^{a-2}(\beta-\alpha)\, \eta^{2a-2}$$
$$+ \; O(\eta^{3a-4}) + O(\eta^{a+n-1}). \qquad \text{(A3.4)}$$

Also on $\xi = h(\eta)$,

$$\theta = \tan^{-1}\{-\cot(\beta-\alpha) - \cot^2(\beta-\alpha)\, c_m \eta^{a-2} + O(\eta^{n-1})\}$$
$$\Rightarrow \theta = -\frac{\pi}{2} - \alpha+\beta - \cos^2(\beta-\alpha)\, c_m\, \eta^{a-2} + O(\eta^{n-1}) + O(\eta^{2a-4}). \qquad \text{(A3.5)}$$

Hence, since $\cos a\beta = 0$,

$$\cos a(\tfrac{\pi}{2} + \alpha + \theta) = a \cos^2(\beta-\alpha)\, c_m\, \eta^{a-2} + O(\eta^{n-1}) + O(\eta^{2a-4}) \qquad \text{(A3.6)}$$

and $$\sin a(\tfrac{\pi}{2} + \alpha + \theta) = 1 + O(\eta^{2a-4}) + O(\eta^{a+n-3}). \qquad \text{(A3.7)}$$

Thus,

$$r^a \cos a(\tfrac{\pi}{2} + \alpha + \theta) = a \sec^{a-2}(\beta-\alpha)\, c_m\, \eta^{2a-2}$$
$$+ \; O(\eta^{3a-4}) + O(\eta^{a+n-1}).$$

Hence from (A3.1), on $\xi = h(\eta)$,

$$\chi = -\tfrac{1}{2}(\eta^2 \sec^2(\beta-\alpha) - 2c_m \tan(\beta-\alpha)\, \eta^a + c_m^2\, \eta^{2a-2}$$
$$- 2c_n \tan(\beta-\alpha)\, \eta^{n+1}) + D_1\, a \sec^{a-2}(\beta-\alpha)\, c_m\, \eta^{2a-2}$$
$$+ D_2\, \eta^b \sec^b(\beta-\alpha) \cos b\beta + o(\eta^{n+1}) + O(\eta^{a+n-1})$$
$$+ O(\eta^{3a-4}) + o(\eta^b). \qquad \text{(A3.8)}$$

As before, $s = \int_0^\eta (1 + (h'(\eta))^2)^{\frac{1}{2}}\, d\eta$.

From (A3.2), $$h'(\eta) = -\tan(\beta-\alpha) + c_m(a-1)\, \eta^{a-2}$$
$$+ n\, c_n\, \eta^{n-1} + o(\eta^{n-1})$$

$$\Rightarrow \{1 + (h'(\eta))^2\}^{\frac{1}{2}} = \sec(\beta-\alpha)\,\{\, 1 - \frac{(a-1)\, c_m \tan(\beta-\alpha)}{\sec^2(\beta-\alpha)}\, \eta^{a-2}$$
$$- \frac{n\, c_n \tan(\beta-\alpha)}{\sec^2(\beta-\alpha)}\, \eta^{n-1} + \frac{(a-1)^2 c_m^2}{2\sec^4(\beta-\alpha)}\, \eta^{2a-4}$$
$$+ \; o(\eta^{n-1}) + O(\eta^{a+n-3})\}.$$

=>/

$$=> \quad s^2 = \eta^2 \sec^2(\beta-\alpha) - 2c_m \tan(\beta-\alpha)\, \eta^a - 2c_n \tan(\beta-\alpha)\, \eta^{n+1}$$

$$+ \frac{c_m^2}{\sec^2(\beta-\alpha)} \{ \tan^2(\beta-\alpha) + \frac{(a-1)^2}{2a-3} \} \eta^{2a-2}$$

$$+ o(\eta^{n+1}) + O(\eta^{a+n-1}). \qquad (A3.9)$$

In order to satisfy (4.2.6) we must have, using (A3.8) and (A3.9),

$$-\tfrac{1}{2} \left(\eta^2 \sec^2(\beta-\alpha) - 2c_m \tan(\beta-\alpha)\, \eta^a + c_m^2\, \eta^{2a-2} - 2c_n \tan(\beta-\alpha)\, \eta^{n+1}\right)$$

$$+ D_1\, a \sec^{a-2}(\beta-\alpha)\, c_m\, \eta^{2a-2} + D_2\, \eta^b \sec^b(\beta-\alpha) \cos b\beta$$

$$+ o(\eta^{n+1}) + O(\eta^{a+n-1}) + O(\eta^{3a-4}) + o(\eta^b)$$

$$= -\tfrac{1}{2}(\eta^2 \sec^2(\beta-\alpha) - 2c_m \tan(\beta-\alpha)\, \eta^a - 2c_n \tan(\beta-\alpha)\, \eta^{n+1}$$

$$+ \frac{c_m^2}{\sec^2(\beta-\alpha)} \{\tan^2(\beta-\alpha) + \frac{(a-1)^2}{2a-3} \} \eta^{2a-2}) .$$

Either $b = 2a-2$ or $b < 2a-2$ and $\cos b\beta = 0$. (A3.10)

It is not possible to have $b > 2a-2$ since the terms in η^{2a-2} cannot be matched on their own.

Now, from (A3.1) we have

$$\frac{\partial\chi}{\partial\xi} = -\xi + D_1\, a\, r^{a-2} \{ \xi \cos a(\tfrac{\pi}{2} + \alpha + \theta) + \eta \sin a(\tfrac{\pi}{2} + \alpha + \theta)\}$$
$$+ D_2\, b\, r^{b-2} \{ \xi \cos b(\tfrac{\pi}{2} + \alpha + \theta) + \eta \sin b(\tfrac{\pi}{2} + \alpha + \theta)\}$$
$$+ o(r^{b-1}) \qquad (A3.11)$$

$$\frac{\partial\chi}{\partial\eta} = -\eta + D_1\, a\, r^{a-2} \{ \eta \cos a(\tfrac{\pi}{2} + \alpha + \theta) - \xi \sin a(\tfrac{\pi}{2} + \alpha + \theta)\}$$
$$+ D_2\, b\, r^{b-2} \{ \eta \cos b(\tfrac{\pi}{2} + \alpha + \theta) - \xi \sin b(\tfrac{\pi}{2} + \alpha + \theta)\}$$
$$+ o(r^{b-1}) . \qquad (A3.12)$$

It follows from (A3.3) that, on $\xi = h(\eta)$,

$$r^{a-2} = \eta^{a-2} \sec^{a-2}(\beta-\alpha) - (a-2)\, c_m \tan(\beta-\alpha) \sec^{a-4}(\beta-\alpha)\, \eta^{2a-4}$$
$$+ O(\eta^{3a-6}) + O(\eta^{a+n-3}). \qquad (A3.13)$$

From/

From (A3.2), (A3.6) and (A3.7) we have

$$\xi \cos a(\tfrac{\pi}{2} + \alpha + \theta) = -a\, c_m \cos^2(\beta-\alpha) \tan(\beta-\alpha)\, \eta^{a-1} + O(\eta^n) + O(\eta^{2a-3}) \qquad (A3.14)$$

$$\eta \sin a(\tfrac{\pi}{2} + \alpha + \theta) = \eta + O(\eta^{2a-3}) + O(\eta^{a+n-2}) \qquad (A3.15)$$

$$\eta \cos a(\tfrac{\pi}{2} + \alpha + \theta) = a\, c_m \cos^2(\beta-\alpha)\, \eta^{a-1} + O(\eta^n) + O(\eta^{2a-3}) \qquad (A3.16)$$

$$\xi \sin a(\tfrac{\pi}{2} + \alpha + \theta) = -\eta \tan(\beta-\alpha) + c_m\, \eta^{a-1} + O(\eta^{2a-3}) + O(\eta^n). \qquad (A3.17)$$

From (A3.13) - (A3.15), on $\xi = h(\eta)$,

$$r^{a-2} \{\xi \cos a(\tfrac{\pi}{2} + \alpha + \theta) + \eta \sin a(\tfrac{\pi}{2} + \alpha + \theta)\}$$
$$= \eta^{a-1} \sec^{a-2}(\beta-\alpha) - 2(a-1)\, c_m \sec^{a-4}(\beta-\alpha) \tan(\beta-\alpha)\, \eta^{2a-3} + O(\eta^{3a-5}) + O(\eta^{a+n-2}).$$

Hence, on $\xi = h(\eta)$, using (A3.11) we find

$$\frac{\partial\chi}{\partial\xi} = \eta \tan(\beta-\alpha) - c_m\, \eta^{a-1} - c_n\, \eta^n + D_1\, a\, \{\eta^{a-1} \sec^{a-2}(\beta-\alpha) - 2(a-1)\, c_m \sec^{a-4}(\beta-\alpha)\tan(\beta-\alpha)\eta^{2a-3} + D_2\, b\, \eta^{b-1} \sec^{b-2}(\beta-\alpha)\, \{-\tan(\beta-\alpha) \cos b\beta + \sin b\beta\} + o(\eta^n) + o(\eta^{b-1}) + O(\eta^{a+n-2}) + O(\eta^{3a-5}). \qquad (A3.18)$$

Again, using (A3.13), (A3.16), (A3.17),

$$r^{a-2}(\eta \cos a(\tfrac{\pi}{2} + \alpha + \theta) - \xi \sin a(\tfrac{\pi}{2} + \alpha + \theta))$$
$$= \eta^{a-1} \sec^{a-2}(\beta-\alpha) \tan(\beta-\alpha) + c_m \sec^{a-4}(\beta-\alpha)\, \eta^{2a-3} (\tan^2(\beta-\alpha)-1)(1-a) + O(\eta^{3a-5}) + O(\eta^{a+n-2}).$$

Hence, on $\xi = h(\eta)$, using (A3.12) we have

$$\frac{\partial\chi}{\partial\eta} = -\eta + D_1\, a\{\eta^{a-1} \sec^{a-2}(\beta-\alpha) \tan(\beta-\alpha) - (a-1)\, c_m \sec^{a-4}(\beta-\alpha)\, (\tan^2(\beta-\alpha)-1)\, \eta^{2a-3}\} + D_2\, b\, \eta^{b-1} \sec^{b-2}(\beta-\alpha)(\cos b\beta + \tan(\beta-\alpha) \sin b\beta) + o(\eta^{b-1}) + O(\eta^{3a-5}) + O(\eta^{a+n-2}).$$

Since/

Since $h'(\eta) = -\tan(\beta-\alpha) + (a-1)\, c_m\, \eta^{a-2} + n\, c_n\, \eta^{n-1}$
$+\ o(\eta^{n-1}),$

$$h'(\eta)\frac{\partial\chi}{\partial\eta} = \eta\tan(\beta-\alpha) - D_1 a\eta^{a-1}\sec^{a-2}(\beta-\alpha)\tan^2(\beta-\alpha) - (a-1)\, c_m\, \eta^{a-1}$$
$$- nc_n\, \eta^n - D_2 b\eta^{b-1}\sec^{b-2}(\beta-\alpha)\tan(\beta-\alpha)$$
$$(\cos b\beta + \tan(\beta-\alpha)\sin b\beta)$$
$$+ 2D_1\, a(a-1)\, c_m \sec^{a-4}(\beta-\alpha)\tan^3(\beta-\alpha)\,\eta^{2a-3}$$
$$+ O(\eta^{3a-5}) + O(\eta^{a+n-2}) + o(\eta^{b-1}) + o(\eta^n). \tag{A3.19}$$

Now (A3.18) and (A3.19) may be used to satisfy (4.2.7). Thus

$$\eta\tan(\beta-\alpha) - c_m\eta^{a-1} - c_n\eta^n + D_1 a\eta^{a-1}\sec^{a-2}(\beta-\alpha)$$
$$- 2D_1 a(a-1)\, c_m \sec^{a-4}(\beta-\alpha)\tan(\beta-\alpha)\,\eta^{2a-3}$$
$$+ D_2 b\eta^{b-1}\sec^{b-2}(\beta-\alpha)\,\{-\tan(\beta-\alpha)\cos b\beta + \sin b\beta\}$$
$$+ o(\eta^n) + o(\eta^{b-1}) + O(\eta^{a+n-2}) + O(\eta^{3a-5})$$

$$= \eta\tan(\beta-\alpha) - D_1 a\eta^{a-1}\sec^{a-2}(\beta-\alpha)\tan^2(\beta-\alpha)$$
$$-(a-1)\, c_m\eta^{a-1} - n\, c_n\, \eta^n$$
$$+ 2D_1 a(a-1)\, c_m\sec^{a-4}(\beta-\alpha)\tan^3(\beta-\alpha)\,\eta^{2a-3}$$
$$- D_2 b\eta^{b-1}\sec^{b-2}(\beta-\alpha)\tan(\beta-\alpha)\,\{\cos b\beta + \tan(\beta-\alpha)\sin b\beta\}. \tag{A3.20}$$

The terms in η^{a-1} already match since $c_m = -D_1 a \sec^a(\beta-\alpha)/(a-2)$. As regards the relative sizes of a, b and n there are various possibilities.

(I) $b-1 = 2a-3<n$

(II) $b-1 = n<2a-3$

(III) $2a-3 = n<b-1$

(IV) $b-1 = 2a-3 = n$.

However/

However it follows from (A3.10) that $b \leqslant 2a-2$ and so (III) is ruled out. Consider case (I). Then (A3.20) implies

$$2D_1 a(a-1)\, c_m \tan(\beta-\alpha) \sec^{a-2}(\beta-\alpha) = D_2 b \sec^b(\beta-\alpha) \sin b\beta$$

$$\Rightarrow D_2 \sec^{2a}(\beta-\alpha) \sin 2\beta = -(a-2)\, c_m^2 \tan(\beta-\alpha) \tag{A3.21}$$

since $D_1 = -(a-2)\, c_m \cos^a(\beta-\alpha)/a$.

But (A3.10) gives

$$-\tfrac{1}{2} c_m^2 + D_1 a \sec^{a-2}(\beta-\alpha)\, c_m + D_2 \sec^b(\beta-\alpha) \cos b\beta$$

$$= -\tfrac{1}{2} \frac{c_m^2}{\sec^2(\beta-\alpha)} \left(\tan^2(\beta-\alpha) + \frac{(a-1)^2}{2a-3} \right)$$

$$\Rightarrow D_2 \sec^{2a}(\beta-\alpha) \cos 2\beta = -\tfrac{1}{2} c_m^2 \, \frac{(2a-3)^2 - (a-1)^2}{2a-3} .$$

But from (A3.21), $D_2 \sec^{2a}(\beta-\alpha) \sin 2\beta = -(a-2)\, c_m^2 \tan(\beta-\alpha)$, and so

$$\tan 2\beta = \frac{2(2a-3)}{3a-4} \tan(\beta-\alpha).$$

Since $a > 2$, $(2a-3)/(3a-4) > 0$ and since $0 < \beta < \pi/4$, $\tan 2\beta > 0$. However when $\alpha > \pi/4$, $\beta - \alpha < 0$ and so $\tan(\beta-\alpha) < 0$. In this case the equality cannot be satisfied and this possibility must be rejected.

Consider case (II). Then since $b < 2a-2$, it follows from (A3.10) that $\cos b\beta = 0$.

Hence $b\beta = 3\pi/2$ since $b > a$, and $b = 3a$.

Since $b < 2a - 2$, $a < -2$ which is impossible.

This case must also be rejected.

There/

There remains case (IV) in which $b-1 = 2a-3 = n$. Then (A3.10) implies

$$D_2 \sec^{2a}(\beta-\alpha) \cos 2\beta = -\tfrac{1}{2} c_m^2 \frac{(3a-4)(a-2)}{(2a-3)} . \quad \text{(A3.22)}$$

Also (A3.20) implies

$$(n-1) c_n + D_2 b \sec^b(\beta-\alpha) \sin b\beta$$
$$= 2D_1 a(a-1) c_m \sec^{a-2}(\beta-\alpha) \tan(\beta-\alpha)$$
$$\Rightarrow c_n = \frac{c_m^2 \cos^2(\beta-\alpha)}{2} \left\{\frac{(a-1)(3a-4)}{(2a-3)} \tan 2\beta - 2(a-1)\tan(\beta-\alpha)\right\} . \quad \text{(A3.23)}$$

We have found that, near the contact point,

$$\chi = -\tfrac{1}{2} r^2 + D_1 r^a \cos a(\tfrac{\pi}{2} + \alpha + \theta)$$
$$+ D_2 r^{2a-2} \cos(2a-2)(\tfrac{\pi}{2} + \alpha + \theta) + o(r^{2a-2})$$

and $h(\eta) = -\eta \tan(\beta-\alpha) + c_m \eta^{a-1} + c_n \eta^{2a-3} + o(\eta^{2a-3})$,

where D_2 and c_n are given by (A3.22) and (A3.23) in terms of c_m and hence of D_1 and where $a = \pi/2\beta$.

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